Proof

Define an inner product on $\mathbb{C}^{2n}$ with orthonormal basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$

Then define a map $\mathbb{C}^{2n} \longrightarrow \mathbb{C}^2$

$e_i \longmapsto e$

$f_i \longmapsto f$

Then $\mathcal{F}l_n^{X} \subset \mathcal{F}l_n \longrightarrow$ decomp of $\mathbb{C}^{2n}$ into n orthogonal lines $\longrightarrow$ collections of lines in $\mathbb{C}^2$

$\mathcal{F}l_n^X \longrightarrow \{\ \} \longrightarrow (L_1, \ldots, L_{2n})$ $\square$.

Corollary

If U, V are two matchings then $\overline{Y_U} \cap \overline{Y_V} \underset{\text{homeo}}{\cong} (\mathbb{P}^1)^k$ where $k = \#$ of circles in

U

V

Now our goal is to prove the following result.

Theorem

There exists an action of S_n on $H_\bullet(\mathcal{F}l_n^X)$.

Moreover $\bullet\ H_N(\mathcal{F}l_n^X) = V_\lambda$ $\qquad N = 2\sum_i (i-1)\lambda_i$

Here V_λ = irreducible representation of S_n associated to the partition λ.

It is known that $\dim V_\lambda = \#\, SYT(\lambda)$

$\bullet\ H_\bullet(\mathcal{F}l_n^X) = \mathbb{C}\begin{bmatrix}\text{row-strict} \\ \text{tableaux}\end{bmatrix}$ "easy representation of S_n"

$= \mathbb{C}[S_n / S_\lambda]$ $\qquad S_\lambda = S_{\lambda_1} \times S_{\lambda_2} \times \cdots$

let $K_{\mu\lambda}(q) = \sum_i \dim \operatorname{Hom}(V_\mu, H_{N-2i}(\mathcal{F}l_n^X))\, q^i$ ~~Kostka~~-Foulkes polynomial

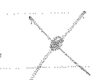
<u>eg</u>

$\lambda = (n)$ $\quad Fl_n^X = pt \quad H_\bullet(Fl_n^X) = \mathbb{C}$ trivial rep of S_n

$\lambda = (1,\dots,1)$ $\quad Fl_n^X = G/B \quad H^\bullet(Fl_n) \cong \mathbb{C}[x_1,\dots,x_n] \big/ \langle \mathbb{C}[x_1,\dots,x_n]_+^{S_n} \rangle = \mathbb{C}[x_1,\dots,x_n] / e_1,\dots,$

and $H^\bullet(Fl_n) = \mathbb{C}[S_n]$ as an S_n-rep

e_i = elementary symmetric function

$\lambda = (2,1)$ $\quad Fl_n^X =$  $\quad H_2(Fl_n^X)$ - 2 dim

$H_0(Fl_n^X)$ - 1 dim

$H_\bullet(Fl_n^X) = \mathbb{C}^n$ as S_n-rep

$H_2(Fl_n^X) = \{(x_1,\dots,x_n) : x_1 + \dots + x_n = 0\}$

<u>So $K_{(1^3),(2,1)} = q$</u>

There are many ways to construct the S_n action on $H_\bullet(Fl_n^X)$.

① First: $H^\bullet(Fl_n)$ has an S_n-action

(i) $H^\bullet(Fl_n)$ = coinvariant alg.

(ii) $Fl_n = K/T$ = orthogonal lines in $\mathbb{C}^n \circlearrowleft S_n$

Next, $H^\bullet(Fl_n) \to H^\bullet(Fl_n^X)$ by restriction

This map is surjective and the kernel is S_n-invariant

② we will pursue a different approach which involves studying the Steinberg variety

$Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$ = union of the conormal bundles to the schubert cells.

$= \{(X, V_\bullet, V'_\bullet) : XV_i \subset V_i,\ XV'_i \subset V'_i\}$

later.

To understand $H^{\bullet}(Fl_n) = \mathbb{C}[x_1 \ldots x_n] / \langle \mathbb{C}[x_1 \ldots x_n]^{S_n}_+ \rangle$

we let $x_i = c_1(V_i / V_{i-1})$

Then $x_1 + \ldots + x_n = c_1(\mathbb{C}^n) = 0$

in fact every $e_i \in \mathbb{C}[x_1, \ldots, x_n]^{S_n}_+$ is the ith Chern class of $\mathbb{C}^n$, thus 0.

Note $H^{\bullet}(Fl_n) = \mathcal{O}\left(\begin{smallmatrix}\text{diagonal}\\ \text{matrices}\end{smallmatrix}\right) \otimes_{\mathcal{O}(M_n)} \mathcal{O}(N) = \mathcal{O}(\mathfrak{t} \cap N)$

$\mathcal{O}(N) =$ coeff of char poly vanish

If $X, Y \subset Z$ two subvarieties

then $\mathcal{O}(X \cap Y) = \mathcal{O}(X) \otimes_{\mathcal{O}(Z)} \mathcal{O}(Y)$

is their scheme theoretic intersection

Eg $X =$  $Y =$

$\mathcal{O}(X) = \mathbb{C}[x,y]/y-x^2$ $\mathcal{O}(Y) = \mathbb{C}[x,y]/y$

$\mathcal{O}(X \cap Y) = \mathbb{C}[x]/x^2$

<u>Theorem</u> (de Concini–Procesi, Tanisaki)

For any λ,

$H^{\bullet}(Fl_n^{\lambda}) = \mathcal{O}(\mathfrak{t} \cap \overline{\mathbb{O}}_{\lambda'})$ as S_n-representations.

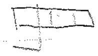
where $\mathbb{O}_{\lambda'}$ = nilpotent matrices of Jordan type λ'

λ' = conjugate partition

eg $n=3$ $\lambda =$  $\lambda' =$ 

$\overline{\mathbb{O}}_{\lambda'} = \{X \in M_n :$ char poly of X vanishes and $X^2 = 0\}$

So $\overline{\mathbb{O}}_{\lambda} \cap \mathfrak{t} = \left\{(x_1, x_2, x_3) : x_i^2 = 0 \;\; \begin{matrix} x_1x_2 + x_1x_3 + x_2x_3 = 0 \\ x_1 + x_2 + x_3 = 0 \end{matrix}\right\}$ $\Rightarrow x_1x_2 = 0$

$\lambda =$  $\lambda' =$  $\overline{\mathbb{O}}_{\lambda}' = \{X \in N, X^2 = 0, \operatorname{rk} X = 1\}$

$\operatorname{tr}(X) = 0$

i.e. all 2×2 minors vanish.

$\mathfrak{t} \cap \overline{\mathbb{O}}_{\lambda}' = \{x_i : x_i^2 = 0 \;\; x_i x_j = 0\}$

$x_1 + \ldots + x_n = 0$

Steinberg variety.

$Z = \{ (X, V_\circ, V_\circ') \; X V \qquad \} = \tilde{N} \times_N \tilde{N}$

= union of conormals to Schubert orbits.

$Y \subset X$ smooth $\quad T^*_Y X = \{ \alpha : \alpha(v) = 0 \;\forall v \in T_y Y \}$

$T^*_Y X$ is half-dimensional Lagrangian $\dim T^*_Y X = \dim X$

$\alpha \in T^*_{V_\circ^1, V_\circ^2}(G/B \times G/B)$ ... really

So $\alpha = (\alpha_1, \alpha_2) \in n(V_\circ^1) \oplus n(V_\circ^2)$

The tagent space to the G-orbit is $g \to g/b(V_\circ^1) \oplus g/b(V_\circ^2)$

So α kills g_{diag} if $\alpha_1 = -\alpha_2$

So $Z \cong$ union of conormals to GL_norbits in $Fl_n \times Fl_n$

Now for any group and subgroup

$H \backslash G/H = {}_G\backslash G/H \times G/H$.

In our case, this gives

$X_w \longleftrightarrow A_w$

$\{ V_\circ : E_\circ$ and $V_\circ$ are in relative position $w \}$ $\qquad \{ V_\circ^1, V_\circ^2 : V_\circ^1$ and $V_\circ^2$ are in rel. position $w \}$

$\overline{T^*_{A_w}(Fl_n \times Fl_n)} = Z_w \subset Z$ an irreducible component

each $T^*_{A_w}(Fl_n \times Fl_n)$ has dimension $2 \dim Fl_n = n(n-1)$

$Z = \bigcup T^*_{A_w}(Fl_n \times Fl_n)$

If U is a standard Young tableau of shape λ,
we have $Y_U \subset Fl_n^X$

We can take $GY_U = \{(X, V_0) : X \in \mathcal{O}_\lambda$ and $X|_{V_i}$ has Jordan type $\{\ldots\}\}$

So $GY_U \subset \pi^{-1}(\mathcal{O}_\lambda)$

$$\dim GY_U = \sum_i (i-1)\lambda_i + \dim \mathcal{O}_\lambda = \sum_i (i-1)\lambda_i + n^2 - \sum (\lambda_i')^2$$

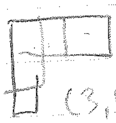
$$\dim \mathcal{O}_\lambda = n^2 - \dim \mathrm{Stab}_{GL_n}(X) = n^2 - \sum (\lambda_i')^2$$

(eg $X = 0$, $\lambda = (1, \ldots, 1)$, $\dim \mathcal{O}_\lambda = n^2 - n^2 = 0$; $\lambda = (n)$, $\dim \mathcal{O}_\lambda = n^2 - n$)

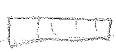
Now take two tableau U, U' of same shape
and form $GY_U \times_{\mathcal{O}_\lambda} GY_{U'} \subset Z$

$$\dim (GY_U \times_{\mathcal{O}_\lambda} GY_{U'}) = 2 \sum_i (i-1)\lambda_i + n^2 - \sum (\lambda_i')^2$$
$$= n(n-1)$$

$n(n-1) = 30$.

 $(3,1,1)$ $\quad 2(1+2) + 25 - (9+1+1) = 30$

 $\quad 2(1) + 36 - (4+4+1+1) = 30$

 $\quad 0 + 36 - (1+\ldots+1) = 30$

 $\quad 2(1+\ldots+5) + 36 - 36 = 30$

0	1
2	3
4	5
6	7

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To see this $GY_U \to Fl_n$ fibre bundle
with fibre having

So for every pair of tableaux U, U'

: $\overline{GY_U \times GY_{U'}}$ is a irreducible component

thus it must equal Z_w for some $w \in S_n$.

$$S_n \longrightarrow \bigcup_\lambda \operatorname{SYT}(\lambda)^2$$

Theorem

If w and (U, U') correspond under the Robinson-Schensted correspondence, then

$$\overline{GY_U \times_{U_\lambda} GY_{U'}} = Z_w.$$

Now $Z = \bigcup_{w \in S_n} Z_w$ is a reducible variety

with $n!$ components each of the same dimension $n(n-1)$

Let $H(Z) := H^{BM}_{2n(n-1)}(Z, \mathbb{C})$ be the top Borel-Moore homology of Z

(BM homology is like ordinary homology, except you allow infinite chains

$H_1^{BM}(\mathbb{R}) = \mathbb{Z}$ $\quad H_0^{BM}(\mathbb{R}) = 0$)

every irreducible variety X has a class $[X] \in H_{2\dim}(X)$

Theorem

$H(Z)$ has an algebra structure, and $H(Z) \cong \mathbb{C}S_n$.

Proof

Given a classes $a, b \in H(Z)$, we can ~~pull back to~~ form the convolution.

$N = 2\dim Fl_n = n(n-1)$

$$\begin{array}{c} \tilde{N} \times \tilde{N} \times \tilde{N} \\ \swarrow^{\pi_{12}} \quad \downarrow^{\pi_{23}} \quad \searrow^{\pi_{13}} \\ a \in \tilde{N} \times \tilde{N} \qquad \tilde{N} \times \tilde{N} \qquad \tilde{N} \times \tilde{N} \end{array}$$

$$a \cdot b := \pi_{13*}\left(\pi_{12}^*(a) \cap \pi_{23}^*(b)\right)$$

H_{4N} $\quad$ H_{4N}

H_{4N}

here $\pi_{12}^*(a)$ mean that if $a=[A]$,
then $\pi_{12}^*(a) = [\pi_{12}^{-1}(A)]$
and $\pi_{12}^*(a) \cap \pi_{23}^*(b)$ means you intersect transversely inside $\tilde{N}\times\tilde{N}\times\tilde{N}$ (smooth)
but resulting in a class supported on $\tilde{N}\times_{N}\tilde{N}\times_{N}\tilde{N}$.

This defines the algebra structure on $H(Z)$

To show that $H(Z) = \mathbb{C}S_n$, we consider

$$\begin{array}{ccccc} \mathbb{C}^n & \xleftarrow{\sim} & \tilde{g} & \longrightarrow & M_n \\ \cup & & \cup & & \cup \\ 0 & \longleftarrow & \tilde{N} & \longrightarrow & N \end{array}$$

recall that if we take $\underline{a} = (a_1, \dots, a_n) \in \mathbb{C}^n$ distinct,
then $\tilde{g}_{\underline{a}} \longrightarrow (M_n)_{\underline{a}}$ is a principal S_n bundle

let $\tilde{g}_{\underline{a}} = \{(X, V_\cdot) : X$ has eig val $a_1, \dots, a_n$, $XV_i \subset V_i\}$

then $S_n \subset \tilde{g}^{rs}$ and $\tilde{g}_{\underline{a}} \times_{(M_n)_{\underline{a}}} \tilde{g}_{\underline{a}} = \bigsqcup_{w \in S_n} \Gamma_w$

We can define in a similar way an algebra structure on $H(Z_{\underline{a}})$ where $Z_{\underline{a}} = \tilde{g}_{\underline{a}} \times_{(M_n)_{\underline{a}}} \tilde{g}_{\underline{a}}$

and $[\Gamma_w] \cdot [\Gamma_{w'}] = [\Gamma_{ww'}]$

Now we have a specialization map $sp: H(Z_{\underline{a}}) \to H(Z)$
which is an isomorphism

(this gives a basis $sp([\Gamma_w])$ for $H(Z)$ which is not the same as the basis $[Z_w]$)

Eg $n=2$ $\quad Z = \{(X, L, L') : XL \subset 0 \quad XL' \subset 0\}$
$= \{\mathbb{C}^2 \supset L, L' \ni 0\} \cup \{\mathbb{C}^2 \supset L = L' \ni 0\}$ ($\mathbb{P}^1 \times \mathbb{P}^1$, Z_s)

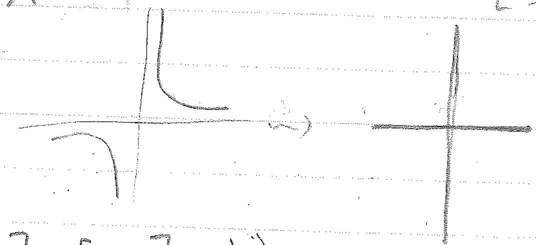
$[Z_s] \cdot [Z_s] = \{\mathbb{C}^2 \supset L \times L' \times L'' \ni 0\} = 2[Z_s]$ ($\mathbb{P}^1$-fibre, $X=0$)

$[Z_s] = 1 - s$
and $\mathbb{C}^2 \supset L \times L'$ disj to union

$(1-s)^2 = 2(1-s)$

$\Gamma_s = \{ (L, L', a) : \text{Eigenval}(X) = \pm a$, $L = a$ eigenspace, $L' = -a$ eigenspace $\}$
X

$\overline{\Gamma_s}\big|_0 = Z$



$= \{ ([1,x], [1,y], \lambda)$

$\begin{bmatrix} a & b \\ c & -a \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} = \begin{bmatrix} a+bx \\ c-ax \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ x \end{bmatrix}$ $\quad a + bx = \lambda$, $\quad c - ax = \lambda x$

$a + by = -\lambda$

$c - ay = -\lambda y$ $\}$

$= \{ x, y, \lambda$; b, c, d : $\boxed{c - x(\lambda - bx) = \lambda x}$ $\Leftrightarrow$ $c = (\lambda + a)x$

$-\lambda - by = \lambda + bx$ $\quad = (-\lambda + a) y$ ✓

$\boxed{2\lambda = (y-x) b}$

$c - (\cancel{\lambda} - b\cancel{x}) y = -\lambda y$ $\quad c = (-by)x = (-bx)y$ ✓

$\{ (x, y, \lambda, b) : 2\lambda = (y-x) b \}$

So if we take $\lambda = 0$ we get

$\{ (x, y, \lambda, b) : (y - x) b = 0 \}$

thus two components $x = y$ and $b = 0$ ✓

Now $[Z_s] \cdot [Z_s]$ carefully

$$\left\{ \begin{tikzcd}[column sep=small] & \mathbb{C}^2 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\ L \arrow[dr, dash] & L' \arrow[d, dash] & L'' \arrow[dl, dash] \\ & 0 & \end{tikzcd} \right\}$$ $x = 0$

we have an excess intersection bundle

dim $= 3$ $\quad$ excess bundle $= T_{\mathbb{P}^1}$ pull-back from second factor

dim $Z_s = 2$ $\quad = \mathcal{O}(2)$ along second factor.

$e(V) \cdot [\;]$ $\in H^2$, $\in H_6$, $\in H_4$ $\rightsquigarrow$ get $=$ tangent bundle of fibre

The resulting basis $[Z_w]$ for $\mathbb{C}S_n$ is not the Kazhdan-Lusztig basis.

The reason is that if T_w denote the KL basis,

then $C_w = \sum_{v \le w} \chi(i_v^* IC_{X_w})\, v$.

$$K(D^b(Fl_n)) \xrightarrow{\sim} \mathbb{C}S_n$$

$$[IC_{X_w}] \longmapsto C_w$$

$$[\mathbb{C}_{X_w}] \mapsto w$$

Now we have a map, characteristic cycle:

$$cc : K(D^b(Fl_n)) \longrightarrow \underset{\substack{\text{Lagrangian} \\ \text{cycles in } T^*Fl_n}}{\Lambda} = H(Z)$$

and cc is an iso. compatible with previous iso.

However $cc([IC_{X_w}]) \neq [Z_w]$ (Kashiwara-Saito)

why is $cc([\mathbb{C}_{X_w}]) = sp(\Gamma_w)$?