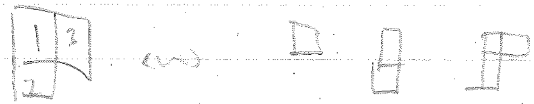
<u>Note</u>: if $X = \begin{bmatrix} 0 & \cdot & \\ & \ddots & \vdots \\ & & 0 \end{bmatrix}$ then $H(X)$ is called the Peterson variety. also $n-1$ dimensional, not smooth or toric.

Then $\lambda^{(k)} \subset \lambda^{(k+1)}$ differ by adding a single box

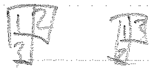
So we get a tableau



For any SYT U, let $Y_U \subset Fl_n^X$ X nilpotent of type λ
of shape λ

$= \{ V_\bullet : (V_\bullet, X) \text{ produce the tableau } \lambda \}$

<u>Eg</u>



<u>Theorem</u>

For each U, $\overline{Y_U}$ is an irreducible component of Fl_n^X, each component has dimension $\sum (i-1)\lambda_i$

This gives a bijection $\mathrm{Irr}(Fl_n^\lambda) \leftrightarrow SYT(\lambda)$

$\dim Fl_n^\lambda = \sum (i-1)\lambda_i$

<u>Proof</u>

We just need to show that Y_U irreducible of maximal dimension.

We define $Y_U \longrightarrow G(n-1, n)$
by $V_\bullet \longmapsto V_{n-1}$

If we take a point $H \in G(n-1,n)$ in the image the fibre will be $Y_{U \setminus \boxed{n}}$.

The image is:

$\{ H \subset \mathbb{C}^n : X\mathbb{C}^n \subset H$ and $X|_H$ has Jordan type $\lambda' \}$

λ' is obtained from λ by deleting the box labelled n

Sps $\boxed{n}$ occurs in the k^{th} column, μ = Jordan type of $X|_H$.

Then X has Jordan type $\lambda \Leftrightarrow \dim \ker X^r = \#$ of boxes in first r columns λ

$\ker X \subset \ker X^2 \subset \cdots \subset \mathbb{C}^n$ $X|_H$ $\mu \Leftrightarrow \dim \ker X_H^r = \#$ of boxes of r col of μ.

So $\mu = \lambda' \Leftrightarrow \dim \ker X^r \cap H = \ker X^r$ for $r = 1, \ldots, k-1$
$\ker X^k \cap H \neq \ker X^k$

$Y = \operatorname{im} X$

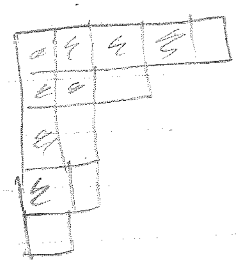
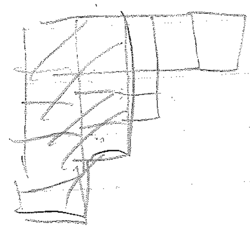
So the image is

$\{H : H \supset \operatorname{im} X + \ker X^{n-1}\} \setminus \{H : H \not\supset \operatorname{im} X + \ker X^{n}\}$

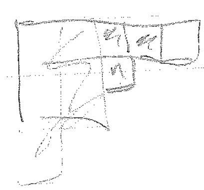
i.e. the image is $\cong \mathbb{P}^\vee\left(\mathbb{C}^n / \operatorname{im} X \cap \ker X^{n-1}\right) \setminus \mathbb{P}^\vee\left(\mathbb{C}^n / \ker X^n\right)$

$$\mathbb{P}(V) \setminus \mathbb{P}(W) \quad W \subset V$$

so the image is smooth of dimension
$n - \dim(\operatorname{im} X \cap \ker X^{n-1})$

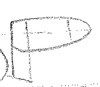
Now  $\ker X^2$ 

$\operatorname{im} X$

$\operatorname{im} X + \ker X^2$ 

So if the n is in the p^{th} row,
$n - \dim(\operatorname{im} X \cap \ker X^{n-1}) = p$. □

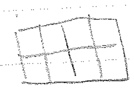
Eg

(1)  has $n-1$ irreducible components each of which $\cong \mathbb{P}^1$

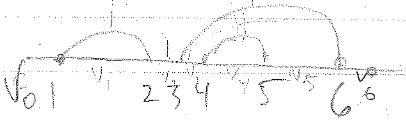
$(n-1,1)$

$\{V_\bullet : V_1 = \langle e_1 \rangle, V_2 = \langle e_1, e_2 \rangle, \dots, V_{i-1} = \langle e_1, \dots, e_{i-1} \rangle$
$V_{i+1} = \langle e_1, \dots, e_i, e_n \rangle \}$

$= \{V_\bullet : V_j = \operatorname{im} X^{n-j-1}$ for $j = 1, \dots, i-1$
$V_j = \ker X^{j+1}$ for $j = i+1, \dots, n\}$

(2)   $\rightsquigarrow$ SYT of shape (n,n) # is $\frac{1}{n+1}\binom{2n}{n}$

Take a matching

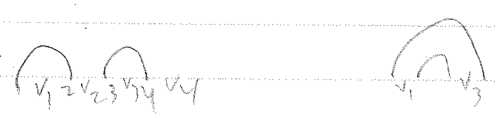


then the component

$\overline{Y_U} = \{ V_\bullet :$ if region i and j are connected, then $V_i = X^{-k} V_j \}$

eg: $V_0 = 0 \subset V_1 \subset V_2 = X^{-1}0 = \ker X \subset V_3 \subset V_4 \subset V_5 \subset V_6 = \mathbb{C}^6$, where $V_5 = X^{-1}V_3$

eg



$V_1 - \langle e_1, e_3 \rangle - V_3 - \mathbb{C}^4$ $\qquad \langle a e_1 + b e_3 \rangle \subset \langle \quad \rangle \subset \langle e_1, e_2, a e_3 + b e_4 \rangle$

e_1	e_2
e_3	e_4

$\mathbb{P}^1 \times \mathbb{P}^1$ $\qquad$ a second Hirzebruch surface

In this 2-row case, every component is smooth.

"Topological Springer fiber"

Consider for each matching U,

$Z_U = \{ (L_i) \in (\mathbb{P}^1)^{2n} : L_i \perp L_j$ if i and j are connected $\}$

<u>Theorem</u>

There is a homeomorphism

$(Fl_{2n})^X \cong \bigcup Z_U \subset (\mathbb{P}^1)^{2n}$ (union of the components)

(L_1, L_2, L_3, L_4) s.t. $L_1 \perp L_2$, $L_3 \perp L_4$

and

(L_1, L_2, L_3, L_4) s.t. $L_1 \perp L_3$, $L_2 \perp L_4$

In particular, each component is homeomorphic to $(\mathbb{P}^1)^n$

they intersect along $(L, L^\perp, L^\perp, L) \cong \mathbb{P}^1$

intersection is # [illegible]