Disussion on toric varieties

$X \subseteq \mathbb{P}(V)$ is called <u>projectively normal</u> if the map $\operatorname{Sym} V^* = \bigoplus_k \Gamma(\mathbb{P}(V), \mathcal{O}(k)) \longrightarrow \Gamma(X, \mathcal{O}(k))$ is surjective

A torus is an alg group T isomorphic to $(\mathbb{C}^\times)^n$ "without basis"

Every torus has a wt lattice $\Lambda = \operatorname{Hom}_{gr}(T, (\mathbb{C}^\times)^n)$

Λ is a free abelian group.

so $\Lambda \cong \mathbb{Z}^n$, not canonically so

choosing $\Lambda \cong \mathbb{Z}^n$ is the same as choosing $T \cong (\mathbb{C}^\times)^n$

A <u>projective toric variety</u> is the following data:

torus T, rep V of T, $X \subseteq \mathbb{P}(V)$ T-invariant, projectively normal, T has a open dense orbit on X.

(I) Every toric variety X determines a polytope $\Phi(X)$.

(i) Choose minimal $W \subset V$ s.t. $X \subset W$, W is a T-invariant subspace

Then take $\Phi(X) := \operatorname{conv}(\mu \in \Lambda : \mu \text{ is a wt of } W)$

So $\Phi(X)$ is a polytope in $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ whose vertices lie in Λ.

(II) Choose a weight basis for V, $V = \mathbb{C}^n$ with wts $\mu_1 \ldots \mu_n$.

Then we have the map

$\Phi : \mathbb{P}(V) \longrightarrow \mathbb{R}^n \longrightarrow \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$.

$[a_1, \ldots, a_n] \mapsto \left(\frac{\|a_1\|}{\|a_1\| + \cdots + \|a_n\|}, \ldots\right)$

$(b_1, \ldots, b_n) \longmapsto \sum b_i \mu_i$

$\Phi(X)$

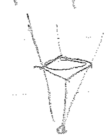
$\mathbb{P}(V) \to \mathbb{R}^n$ is the moment map for $(S^1)^n \curvearrowright \mathbb{P}(V)$

i.e. for any $x \in \mathbb{R}^{n*}$

set $\mathbb{P}(V) \longrightarrow \mathbb{R}^n \longrightarrow \mathbb{R}$

this function gives the [illegible]

② Conversely suppose $P \subset \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ ($= \mathfrak{t}^*_{\mathbb{R}}$) is a polytope whose vertices lie in Λ.



Then form the $C(P) \subset \mathfrak{t}^*_{\mathbb{R}} \oplus \mathbb{R}$

and $C(P) \cap (\Lambda \oplus \mathbb{Z})$ a semigroup, assume generated by $P \cap \Lambda = \{\mu_1, \dots, \mu_n\}$

Then form $\mathbb{C}[C(P) \cap (\Lambda \oplus \mathbb{Z})] = R(P)$ semigroup alg.

We have $\mathbb{C}[z_1, \dots, z_n] \twoheadrightarrow R(P)$

and thus $\mathrm{Proj}(R(P)) \subset \mathbb{P}(\mathbb{C}^n)$ is a projective variety

projective normal by construction.

Then T acts on $\mathbb{C}^n$ by wts $\mu_1, \dots, \mu_n$.

We get a projective toric variety.

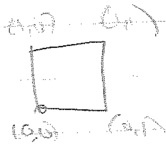
③ Equivalently, given $\mu_1 \dots \mu_n$, define

$T \subset \mathbb{C}^n$ and then pick a point $v = (v_1, \dots, v_n) \in (\mathbb{C}^\times)^n$

Then form $Tv = \{(t^{\mu_1} v_1, \dots, t^{\mu_n} v_n) : t \in T\}$.

Then $\overline{Tv} = X$

Eg



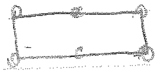
we get the semigroup alg

$$\mathbb{C}[z_1, z_2, z_3, z_4] / z_1 z_4 - z_2 z_3$$

So get $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ Segre embedding

where $(\mathbb{C}^\times)^2$ acts on $\mathbb{C}^4$ by

$$(t_1, t_2) \cdot (v_1, v_2, v_3, v_4) = (v_1, t_1 v_2, t_2 v_3, t_1 t_2 v_4)$$



get $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^5$ in fact $\mathbb{P}(\mathrm{Sym}^2 \mathbb{C}^2 \otimes \mathbb{C}^2)$

$\mathbb{P}(\mathbb{C}^2) \times \mathbb{P}(\mathbb{C}^2) \hookrightarrow$

i.e. ... polytope $\rightleftharpoons$ toric variety with embedding

fan = toric variety

Now let $X \subset \mathbb{P}(V)$ be any proj. variety with $T \circlearrowright V$, X is T invariant.

We can choose any $x \in X$

Then we can form a toric variety $\overline{Tx} \subset X \subset \mathbb{P}(V)$

We say Tx is Tgeneric if whenever $W \subset V$ and $Tx \subset \mathbb{P}(W)$, then $X \subset \mathbb{P}(W)$

In this case $\Phi(Tx) = \Phi(X) = \mathrm{conv}\ V^* \to \Gamma(X, \mathcal{O}(1))$
T-gen

$\Phi(X) = \mathrm{conv}(\text{image of moment map})$

Now we take $Fl_n \subset \mathbb{P}(V(\lambda))$

or $Fl_n \subset \mathbb{P}(\bigotimes_k \mathrm{Sym}^{a_k}(\Lambda^k \mathbb{C}^n))$

Then $\Phi_\lambda(Fl_n) = \mathrm{Conv}(w\lambda : w \in S_n) \subset \mathbb{R}^n$

$Fl_n \cong H_\lambda \xrightarrow{\text{diagonal}} \mathbb{R}^n$

$Fl_n \hookrightarrow \mathbb{P}(V) \to \mathbb{R}^N \to \mathbb{R}^n$

maps are the same since both moment maps for some symplectic form λ

For λ regular, $\lambda_i > \lambda_{i+1}$, this is called the permutahedron.

Lemma TFAE

① $V_\bullet \in Fl_n$ is T-generic iff ② $V_\bullet$ is transverse to every coordinate subspace i.e. $V_\bullet$ and $E_\bullet^w$ are in relative position w_0 for every $w \in S_n$

Given a flag $V_\bullet$, let $b(V_\bullet) = \{X \in \mathrm{End}\, \mathbb{C}^n : XV_i \subseteq V_i\}$

$n(V_\bullet) = \{X \in \quad : XV_i \subseteq V_{i-1}\}$

example $b(E_\bullet) = b = \begin{bmatrix} \diagdown & * \\ 0 & \end{bmatrix}$

$n(E_\bullet) = n = \begin{bmatrix} 0 & * \\ & 0 \end{bmatrix}$

Lemma T

For any $V_\bullet$, we have ident

(i) $T_{V_\bullet}(Fl_n) = M_n / b(V_\bullet)$

(ii) $T^*_{V_\bullet}(Fl_n) = n(E_\bullet)$

Proof

(i) GL_n act trans stabilized

(ii) trace pairing

Lemma

③ $V_\bullet \in Fl_n$ is T-generic iff $b(V_\bullet) \cap t = \mathbb{C}I$

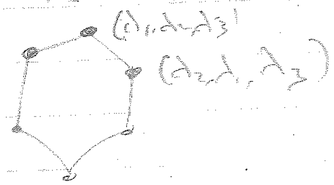
④ $\{t \in T : tV_\bullet = V_\bullet\} = \mathbb{C}^\times I$

Now let us pick a Tgeneric flag $V_\bullet$

$\overline{T V_\bullet}$ is a toric variety corresponding to the permutahedron.

Question
What is $\overline{TV_\bullet}$? is it smooth?

$\overline{T(V_\bullet)}$ is the toric variety associated to the permutohedron (appeared in June Huh's talk)
so it is smooth.



Let $X \in M_n$.

The Hessenberg variety of X is

$H(X) = \{ V_\bullet : XV_i \subset V_{i+1} \}$. Note $T \circlearrowright H(X)$
note $E_\bullet^w \in H(X)$ $\forall w$.

Let $X = \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix}$, $a_i \neq a_j$ $\forall i$, regular semisimple matrix

One way to produce a point in $H(X)$ is as follows

Let $v \in \mathbb{C}^n$ be any vector all coordinates non-zero.
$(v_1, \dots, v_n)$
Form
$V_\bullet = \langle v \rangle \subset \langle v, Xv \rangle \subset \langle v, Xv, X^2 v \rangle \subset \dots$

This is a flag, clearly lives in $H(X)$, and $V_\bullet$ is T-generic.

We have $T = (\mathbb{C}^\times)^n$ acts on $H(X)$.

Lemma
$H(X) = \overline{T(V_\bullet)}$

In fact if $t \in T$, then
$tV_\bullet = (\langle tv \rangle \subset \langle tv, Xtv \rangle \subset \dots)$
... T acts ... by transitively on all

To see the smoothness,
note that
$T_{E_\bullet}(H(X)) = \begin{bmatrix} & * \\ \ddots & \end{bmatrix} / b$

$n-1$ dim. $XV_i \subset V_{i+1}$
... smooth at all $E_\bullet$ ...

Proof

① $T(V_\bullet) = \mathbb{C}^{n-1}$ ② $E_\bullet^w \in T(V_\bullet)$ for all w

③ $\dim T_{E_\bullet^w}(H(Y)) = n-1$ - tangent space in flag variety $\Rightarrow \dim H(Y) \leq n-1$

④ $H(Y)^T \subset Fl_n^T$ so $H(Y)^T = \{E_\bullet^w\}$ $\Rightarrow H(Y) = \overline{T(V_\bullet)}$

⑤ $H(Y)^T$ is smooth at any pt of $H(Y)^T$ $\square$

Now we will begin our study of Springer fibres

We consider $\tilde{\mathfrak{g}} = \{(X, V_\bullet) : X \in M_n, \; V_\bullet \in Fl_n, \; X V_i \subseteq V_i \; (X \in \mathfrak{b}(V_\bullet))\}$

We have

$$\begin{tikzcd} & \tilde{\mathfrak{g}} \arrow[r] \arrow[d] \arrow[dl] & \mathbb{C}^n \arrow[d] \\ Fl_n & \mathfrak{g} \arrow[r] & \mathbb{C}^n/S_n \end{tikzcd}$$

$\mathfrak{g} = M_n$

Also we have $\mathcal{N} \subset \mathfrak{g}$ the nilpotent cone

$\mathcal{N} = \{X \in M_n : X \text{ nilpotent}\}$
$= \{X : \text{characteristic poly of } X \text{ is } 0\}$

$$\tilde{\mathcal{N}} \subset \tilde{\mathfrak{g}}$$

$$\tilde{\mathcal{N}} = \{(X, V_\bullet) : X \in M_n, \; V_\bullet \in Fl_n \; X V_i \subset V_{i-1}\}$$

Lemma
$\tilde{\mathcal{N}} = T^* Fl_n$

Proof
recall that $\mathfrak{n}(V_\bullet) = T^*_{V_\bullet}(Fl_n)$

We see that $X \in \mathfrak{n}(V_\bullet)$ iff $(X, V_\bullet) \in \tilde{\mathcal{N}}$.

$\tilde{\mathfrak{g}}$ is the vector bundle with fibre $\mathfrak{b}(V_\bullet)$
$\tilde{\mathcal{N}}$ $\qquad \mathfrak{n}(V_\bullet)$

For any $X \in \mathcal{N}$, the fibre $Fl_n^X = \{V_\bullet : X V_i \subseteq V_{i-1}\}$ is called the Springer fibre at X.

(it might look wierd, but actually $Fl_n^X = \{V_\bullet : \text{the vector field } X \text{ is } 0\}$
i.e. $X \in \mathfrak{b}(V_\bullet)$)

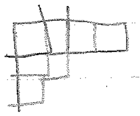
Eg

(1) $X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad Fl^X = \begin{matrix} \langle e_1 \rangle \subset V_2 \\ V_1 \subset \langle e_1, e_3 \rangle \end{matrix} \quad = \quad \cancel{\langle e_1 \rangle \subset \langle e_1, e_3 \rangle}$

(2) $X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad Fl_n^X = \langle e_1 \rangle \subset \langle e_1, e_2 \rangle$

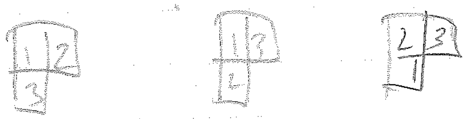
Recall that GL_n orbits on $\mathcal{N} \longleftrightarrow$ partitions of n

A matrix X has "Jordan type λ" if when we write it in Jordan canonical form the blocks have sizes $\lambda_1, \dots, \lambda_k$.



note that you can read off a lot from partition
kernels of powers   images of powers, ...

A row-strict tableau of shape λ is a filling of λ by $1, \dots, n$ increasing in rows.



called standard if also column strict

Theorem
Every rowstrict tableau U determines a point $E_\bullet^U \in Fl_n^X$

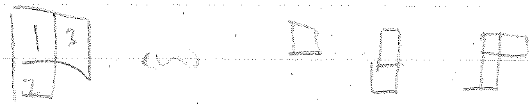
In fact we have $(Fl_n^X)^T = \{ E_\bullet^U : U \text{ rowstrict} \}$
where $T \subset (\mathbb{C}^\times)^n$ is the stabilizer of X.
$\cong (\mathbb{C}^\times)^{\text{number of rows of } \lambda - 1}$

Let $(X, V_\bullet) \in \tilde{\mathcal{N}}$, then we can produce a standard Young tableau as follows

Let $\lambda^{(1)}$ = Jordan type of $X|_{V_1}$, $\lambda^{(2)}$ = Jordan type of $X|_{V_2}$ ...

Then $\lambda^{(k)} \subset \lambda^{(k+1)}$ differ by adding a single box

So we get a tableau



For any SYT U, let $Y_U \subset Fl_n^X$ X nilpotent of type λ
of shape λ

$= \{ V_\bullet : (V_\bullet, X) \text{ produce the tableau } \lambda \}$

Eg $\begin{smallmatrix} 1 & 2 \\ 3 \end{smallmatrix}$ $\begin{smallmatrix} 1 & 3 \\ 2 \end{smallmatrix}$

Theorem
For each U, $\overline{Y_U}$ is an irreducible component of Fl_n^X

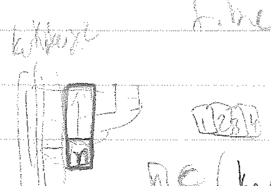
This gives a bijection $\mathrm{Irr}(Fl_n^X) \leftrightarrow SYT(\lambda)$

$\dim Fl_n^\lambda = \sum (i-1)\lambda_i$

Proof
We just need to show that Y_U has maximal dimension.

1	3	4
2		
5		

$Y_U \to G(n-1, n)$ fibre has dimension $k-1$
fibre is $Y_{U \setminus \{\boxed{n}\}}$ if $\boxed{n}$ is on the k^{th} row



$\boxed{n} \in (\ker X^i) \cap (\operatorname{im} X^j)$