ASSIGNMENT 7
DUE THURSDAY MARCH 22

(1) The goal of this exercise is to prove (and then apply) the following result, known as the going-up theorem.

**Theorem 1.** Let $R \subset S$ be two rings such that $S$ is finitely generated as an $R$-module. If $P$ is a prime ideal in $R$, then there exists a prime ideal $Q$ in $S$ such that $Q \cap R = P$.

(Actually the theorem holds (with almost the same proof) in the more general case that $S$ is integral over $R$.)

(a) Assume that $R$ is a local ring and $P$ is its unique maximal ideal. Use Nakayama’s lemma to prove the going-up theorem in this special case.

(b) Use localization at $P$ to deduce the general case of the going-up theorem from the above special case.

(c) Suppose that $X$ and $Y$ are algebraic varieties over an algebraic closed field. Let $\phi : X \to Y$ be a morphism such that $\phi^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ is injective and $\mathcal{O}(X)$ is finitely generated as a $\phi^*(\mathcal{O}(Y))$-module. Use the going-up theorem to prove that $\phi$ is surjective.

(d) Find an example of a morphism $\phi : X \to Y$ such that $\phi^*$ is injective, but $\phi$ is not surjective.

(2) Let $k$ be an algebraically closed field and let $X$ be an affine variety over $k$. Show that giving a $k$-algebra homomorphism $\mathcal{O}(X) \to k[x]/x^2$ is equivalent to giving a point $a \in X$ and an element $v \in T_a X$.

(3) (a) Let $M$ be an $R$-module. Prove that if $M_P = 0$ for all prime ideals $P$, then $M = 0$.

(b) Let $\phi : M \to N$ be a morphism of $R$-modules. Prove that $\phi$ is injective (resp. surjective) if and only if $\phi_P : M_P \to N_P$ is injective (resp. surjective) for all prime ideals $P$.

(4) Let $k$ be a field and consider the ideal $I = \langle xy, y^2 \rangle \subset k[x, y]$. Let $M = k[x, y]/I$. Find $\text{Ass}(M)$ and $\text{Supp}(M)$. 