ASSIGNMENT 6
DUE THURSDAY MARCH 8

(1) Let $f \in k[x]$ and consider $A = k[x, y]/(y^2 - f)$. Assume that $f$ is not a square (note that this ensures that $A$ is a domain). Show that $A$ is integrally closed if and only if $f$ has no square factors. If $A$ is not integrally closed, explain how to find its normalization using $f$.

(2) Let $k \subset L$ be a finite Galois extension and let $G = Gal(L/k)$. Let $I = k[x_1, \ldots, x_n]$ be a radical ideal. Let $R = k[x_1, \ldots, x_n]/I$.

Let $Z(I)_L$ denote the zero set of $I$ inside $k^n_L$. Prove that $G$ acts on $Z(I)_L$ and prove that the orbits of this action are in bijection with the maximal ideals $m$ of $R$ such that $R/m$ is isomorphic to a subfield of $L$. (Hint: start with $n = 1$.)

(3) Define $X$ to be the set of pairs of $n \times n$ matrices whose product in either direction is 0, i.e.

$$X := \{(A, B) \in Mat_{n,n}(k) \times Mat_{n,n}(k) : AB = BA = 0\}$$

Show that $X$ is an affine variety. Find (without proof) the irreducible components of $X$. (Hint: start with $n = 1$.)