1. Find the Galois groups of the polynomial $x^4 - 2$ over each of the fields $\mathbb{Q}$, $\mathbb{F}_3$, and $\mathbb{F}_5$. You may use without proof the following facts:

   - $x^4 - 2$ is irreducible over $\mathbb{Q}$.
   - $x^4 - 2 = (x^2 - x - 1)(x^2 + x - 1)$ over $\mathbb{F}_3$.
   - $x^4 - 2$ is irreducible over $\mathbb{F}_5$.

2. Let $F \subset K$ be a Galois extension with Galois group $G$. Suppose that an intermediate field $F \subset E \subset K$ and a subgroup $H \subset G$ correspond, in the sense that $H = Gal(K/E)$. Prove that $F \subset E$ is a Galois extension if and only if $H$ is a normal subgroup of $G$.

3. Let $R$ be a Noetherian commutative ring and let $M$ be a finitely generated $R$-module. Suppose that $f : M \to M$ is a surjective $R$-module morphism. Prove that $f$ is injective. (You may use the following result: if $M$ is a finitely generated module over a Noetherian ring, then there are no infinite ascending chains of submodules of $M$.)

4. Let $V$ be an irreducible complex representation of a finite group $G$. Let $H \subset G$ be a subgroup of index $k$. Let $W \subset V$ be an $H$-invariant subspace.

   (a) Prove that $\dim W \geq \frac{1}{k} \dim V$.

   (b) Prove that if $\dim W = \frac{1}{k} \dim V$, then $W$ is an irreducible $H$-representation.

5. Let $G$ be a finite group. Prove that the following are equivalent.
(a) For every $g \in G$, there exists $h \in G$ such that $g^{-1} = hgh^{-1}$.
(b) For every complex representation $V$ of $G$, $V \cong V^*$.

6. Let $k$ be an algebraically closed field. Recall the following results.

**Zariski’s Lemma**
If $k \subset F$ is a field extension such that $F$ is finitely generated as a $k$-algebra, then $F = k$.

**Weak form of Hilbert’s Nullstellensatz**
If $I \subset k[x_1, \ldots, x_n]$ is a proper ideal, then $Z(I) \neq \emptyset$.

(a) Prove Zariski’s Lemma using the weak form of the Nullstellensatz.
(b) Prove the weak form of Nullstellensatz using Zariski’s lemma.