Stability and Pinning of Solitons for the NLS

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(Dated: April 2002)

In this expository note, we begin by deriving a Lyapunov Stability result for soliton solutions of the NLS. We then consider the NLS with a potential and demonstrate that soliton like solutions concentrate near the critical point of the potential. The result depends heavily on the Implicit Function theorem and the proof is mostly variational in nature. Some numerical results are also presented at the end.

I. A NOTE ON EXISTENCE

The NLS is given by

$$i\partial_t \psi + \Delta\psi + f(|\psi|^2)\psi = 0$$

with initial condition

$$\psi(x,0) = \psi_0(x)$$

where $x \in \mathbb{R}^n,\ \ t \in \mathbb{R}$. The following Theorem sums up the essential global existence properties of the NLS for solutions with initial data in H^1. We begin by stating some assumptions about our non-linearity, $f(|\psi|^2) := f(|\psi|^2)(x)$.

To ensure that the NLS has a global in time H^1 solution for H^1 data we need the following conditions.

First off we have that $f : [0,\infty) \to \mathbb{R}$ is smooth and satisfies $f(0) = 0$ and

$$|f'(s)| \leq K(1 + s^{\alpha-1}),$$

for some $\alpha \in [0, \frac{2}{d-2})$ when $d \geq 3$. For the case when $d = 1, 2$ then we require that $\alpha \in [0,\infty)$. Also,

$$f(s) \leq K(1 + s^{\beta}),$$

for $\beta \in [0, \frac{2}{d})$.

We can state

Theorem 1 *If we make the above assumptions on* f *, then the NLS has global* H^1 *solutions for small* H^1 *initial data.*

The proof of this theorem is conducted in two steps. First, one proves a local(in time) existence (i.e. on a small time interval). To this end, using the Duhamel Principle one rewrites the NLS as

$$\psi(x,t) = e^{-it\Delta}\psi_0 + i\int\limits_0^t e^{-i(t-s)\Delta}[f(|\psi|^2)\psi(s)]ds,$$

and then applies a fixed point argument to it in an appropriate Banach space. A key point here is that the length of the time interval should depend only on the H^1 norm of ψ.

During the second step, one iterates this procedure with help from estimates that one gets from the conservation of energy. For details check out [G1,V1], [G2,V2], and [Caze] for example as well as a survey in [S,S] and [PEZ].

Section 3 of this work requires that we know a little more, specifically the long-time existence of the flow for the nonlinear Schrodinger equation with potential V on $\mathbb{R}^n$ given by

$$i\frac{\partial\psi}{\partial t} = -\Delta\psi + V\psi - f(|\psi|^2)\psi.$$

This was accomplished by Yong-Geun Oh in [Oh1].

II. ORBITAL STABILITY OF SOLITON SOLUTIONS

Consider the non-linear Schrodinger Equation without potential given by

$$i\partial_t\psi + \Delta\psi + f(|\psi|^2)\psi = 0 \tag{1}$$

with initial conditions

$$\psi(0,x) = \psi_0(x) \quad x \in \mathbb{R}^d$$

and $\psi_0 \in H^1$.

In this section we wish to show that for initial data which is near a ground state soliton solution of (1), the travelling soliton solution will remain near the ground state, where by near we mean close in energy. This is what is referred to as orbital stability. These notions will be made precise as we go on. We begin by exploring the structure of the manifold of ground state solitary waves by examining the symmetries of equation (1). We then make precise the notion of orbital stability which then allows us to state the main theorem of this section. To assist in the proof of the theorem we mention and demonstrate a couple of important conservation laws.

We look for solitary waves of the form $\psi(t,x) = e^{iEt}R(x)$. Substituting this anzatz into equation (1) we arrive at the equation

$$\Delta R - ER + f(|R|^2)R = 0. \tag{2}$$

The first thing to notice before we go on is that equation (1) enjoys gauge and translation symmetries. Indeed, if $\Phi(t,x)$ is a solution to (1) then so is $e^{i\gamma}\Phi(t, x + x_o)$ for $\gamma \in [0, 2\pi)$ and $x_o \in \mathbb{R}^n$. This means that the manifold of ground state solutions is given by

$$M_R = \{e^{i\gamma}R(x + x_o)\big|(x_o, \gamma) \in \mathbb{R}^d \times [0, 2\pi), \}$$

if we also assume that the ground state solitons are spherically symmetric.

Now, we say that the ground state solitary waves of the NLS are orbitally stable if for any $\varepsilon \geq 0$, $\exists\ \delta(\varepsilon) \geq 0$, such that if

$$d(\psi_0(t), M_R) < \delta(\varepsilon), \tag{3}$$

then for all $t > 0$

$$d(\psi(t), M_R) < \varepsilon. \tag{4}$$

Here d is an appropriate metric which measures the distance from the solution ψ to the manifold M_R. In our case d is simply the H^1 norm.

To reiterate, our goal for this section is to outline a proof which demonstrates that, under appropriate conditions, the ground state solitary waves of the NLS are orbitally stable. Now, if $f(|R|^2)$ satisfies the conditions in Section 1 we can state

Theorem 2 *Take $\psi(t,x)$ as the unique solution of (1) with initial condition $\psi_0 \in H^1$. Then the ground state is orbitally stable.*

Before outlining an idea of the proof, we should introduce a couple of conservation laws that will be of some use. Assuming that ψ is a solution to (1), by taking a Frechet derivative it can be seen that the energy functional

$$H_0(\psi) = \frac{1}{2}\int (|\nabla\psi|^2 - F(|\psi|^2)) \tag{5}$$

where $F' = f$, is a constant of motion,i.e.

$$\partial_t H_0(\psi) = 0.$$

To see this notice that we can write (1) as

$$\partial_t \psi = J H_0'(\psi)$$

where $J^* = -J$. Then take

$$\frac{\partial H_0}{\partial t} = \langle H_0'(\psi), \partial_t \psi \rangle = \langle H_0'(\psi), J H_0'(\psi) \rangle = 0.$$

We can also see that the so called 'particle number' $N(\psi)$ is also a constant where

$$N(\psi) = \frac{1}{2} \int |\psi|^2. \tag{6}$$

To see this multiply the NLS by $-i\bar{\psi}$, integrate by parts and take the real part. We find

$$Re \int \bar{\psi} \partial_t \psi + i|\psi|^{2\sigma+2} + i\bar{\psi}\Delta\psi = 0$$

which gives us

$$\partial_t \int |\psi|^2 = 0$$

and so $N(\psi)$ is indeed a constant.

Now we introduce the Lyapunov functional

$$H_E = H_0(\psi) + E\ N(\psi).$$

We now know that both H_0 and N are conserved and therefore we have that

$$\partial_t H_E(\psi) = 0.$$

Consider the tangent space to M_R at a point R given by $T_R M = span\{T, G\}$ where

$$T = \partial_{x_o}(R(x + x_o)e^{i\gamma})\Big|_{x_o=0,\ \gamma=0} = \nabla R(x) \tag{7}$$

and

$$G = \partial_{\gamma}(R(x + x_o)e^{i\gamma})\Big|_{x_o=0,\ \gamma=0} = iR(x). \tag{8}$$

Take ψ as

$$\psi = R_{x_o\gamma} + w_{x_o\gamma}$$

where $\nabla R_{x_o\gamma} \in T_R M$ and $\nabla w_{x_o\gamma} \in T_R M^{\perp}$, $T_R M$ the tangent space of M. We have the Taylor expansion

$$H_E(R + w) = H_E(R) + \langle H_E'(R), w \rangle + \frac{1}{2}\langle w, H_E''(R) w \rangle + R(w). \tag{9}$$

Notice that $\langle H_E'(R), w \rangle = 0$ and

Lemma 3 $R(w) = O(\|w\|^3_{H^1})$.

PROOF:

To demonstrate this fact about $R(w)$, define the real valued function $h(t)$ by

$$h(t) = \frac{1}{2}\int(|\nabla(R+tw)|^2 + E|R+tw|^2 - F(|R+tw|^2))$$

and Taylor expand about 0. We find

$$h(1) = H_E(R) + \frac{1}{2}H_E''(R)|w|^2 + \frac{1}{6}H_E'''(R)|w|^3 + \cdots .$$

Looking at the last term of the above we can deduce that

$$|R(w)| \leq B\int |(f'(R^2)R + f''(R^2)R^3|\ |w|^3 \leq M\int |w|^3 = M\|w\|_{L^3}^3,$$

where B is a constant and M is the L^∞ norm of $|f'(R^2)R + f''(R^2)R^3|$. Notice however, that we have the imbedding

$$H^{\frac{d}{6}} \hookrightarrow L^3$$

which means that for dimensions $d \geq 6$ we have the inequality

$$\|w\|_{L^3} \leq \|w\|_{H^1}.$$

This means then that we have

$$R(w) \leq M\|w\|_{H^1}^3$$

as desired. ■

Due to conservation we have

$$\begin{aligned} H_E(\psi_o) = H_E(\psi) &= H_E(R+w) && (10)\\ &= H_E(R) + \frac{1}{2}\langle w, H''(R)w\rangle + R(w). && (11) \end{aligned}$$

Let

$$\Delta H = H_E(\psi_o) - H_E(R) \tag{12}$$

and

$$L := H''(R).$$

Using (11), we can rewrite (12) as

$$\Delta H = \frac{1}{2}\langle w, Lw\rangle + R(w). \tag{13}$$

We need a lemma that we prove later.

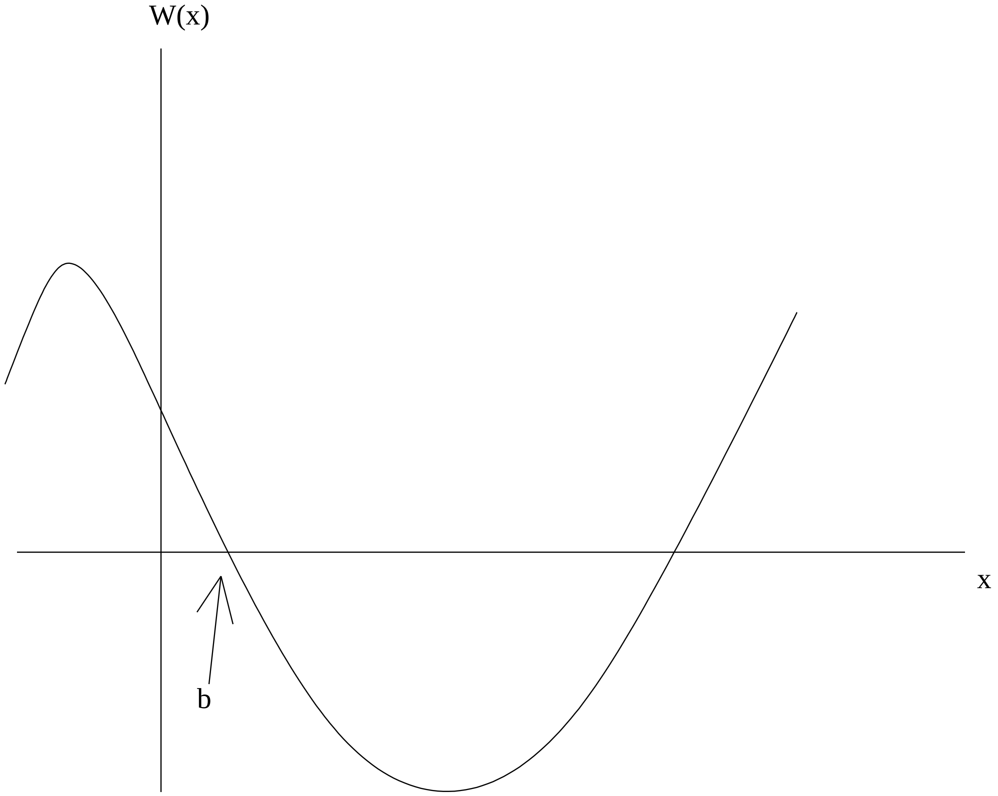

FIG. 1:

Lemma 4

$$\langle w, Lw \rangle \geq \frac{\alpha}{2}\|w\|_{H^1}^2 - c\|w\|_{H^1}^3$$

for $w \perp T_R M$, *provided*

$$\int |\psi|^2 = \int |R|^2.$$

Assuming Lemma 2 we reason as follows. From (13) and Lemma 1, we have

$$\Delta H \geq \frac{\alpha}{4}\|w\|_{H^1}^2 - \frac{c}{2}\|w\|_{H^1}^3.$$

In other words we have the left hand side

$$\frac{c}{2}\|w\|_{H^1}^3 - \frac{\alpha}{4}\|w\|_{H^1}^2 + \Delta H \geq 0.$$

One can see this as a cubic polynomial in $\|w\|_{H^1}$. It is necessary to find a ΔH so that $\|w\|_{H^1}$ is bounded. Since $\|w\|_{H^1}$ can never be negative one just has to ensure that ΔH is small enough so that our polynomial intersects with the $\|w\|_{H^1}$ axis. Let $x = \|w\|_{H^1}$ and

$$W(x) = \frac{c}{2}x^3 - \frac{\alpha}{4}x^2 + \Delta H.$$

One can then refer to figure (1). By considering the critical points of the polynomial one finds that this occurs when

$$\Delta H \leq \frac{5\alpha^3}{108}.$$

When this condition is fulfilled we are guaranteed that

$$\|w\|_{H^1} \leq \beta\sqrt{\frac{\Delta H}{\alpha}},$$

for $\beta \in \mathbb{R}^+$, i.e. in figure (1)

$$b = \beta\sqrt{\frac{\Delta H}{\alpha}}.$$

This gives

$$\|\psi - R_{x_o\gamma}\|_{H^1} = \|w\|_{H^1} \leq \beta\sqrt{\frac{\Delta H}{\alpha}} \tag{14}$$

which ends the argument if we let our metric d be the H^1 norm. ■

How does one go about producing the estimate in Lemma 2? We proceed by first discussing some spectral properties of L. It turns out that L has a negative eigenvalue, but that it will not prevent us from achieving a positive estimate for L.

Recall that L has real and imaginary parts given respectively by L_+ and L_- where

$$L_+ = -\Delta + 1 - (f(R^2) + 2R^2 f'(R^2)) \tag{15}$$

and

$$L_- = -\Delta + 1 - f(R^2). \tag{16}$$

Now, we know something about L_+ and L_- that is very helpful. That is we know that

$$L_+ \frac{\partial R}{\partial x_j} = 0 \quad \& \quad L_- R = 0 \quad ; j = 1, ..., n. \tag{17}$$

To see this just notice that $R \in M_R$ and so by definition it satisfies $L_- R = 0$. Now, differentiating this equation with respect to x_j for $j = 1, ..., n$ gives the other half of (17).

In one dimension we can show that L has one negative eigenvalue as follows. The Sturm-Liouville theorem then tells us, being that $R > 0$, that 0 is the lowest eigenvalue of L_- and that it is simple. In fact, you can be assured that this result holds in all dimensions and so we can say that L_- is a non-negative operator. Also, by the same reasoning and using the fact that $\frac{\partial R}{\partial x_j}$ has one zero, we can see that 0 is the second smallest eigenvalue of L_+. We then see that the spectrum of L_+ contains a negative eigenvalue, and that it is simple. For dimensions 2 and higher, we reason as follows. (This is taken from [MW]. An alternate proof is included in appendix B of [FGJS].)

Proposition 5 *L_+ has exactly one negative eigenvalue.*

Idea of Proof :

We already discussed the situation for the one dimensional case. For higher dimensions we begin by considering the specific non-linearity $f(|\psi|^2) = |\psi|^{2\sigma}$. We show that this L_+ has exactly one negative eigenvalue.

Define the Raleigh quotient J[u] to be the functional

$$J[u] = \frac{\|\nabla u\|_{L^2}^{\sigma d} \|u\|_{L^2}^{2+\sigma(2-d)}}{\|u\|_{2\sigma+2}^{2\sigma+2}}. \tag{18}$$

One could evaluate the Frechet derivative $\delta^2 J[\psi]|_{\psi=R}$ and find that

$$\langle L_+ h, h\rangle + k^2(\sigma d - 2)\langle h, \Delta R\rangle^2 \geq 0,$$

for functions h such that $\langle h, R\rangle = 0$. This means that we have a non-negative operator $L_+ + U$ where U is an operator of rank one. Thus, L_+ can have at most one negative eigenvalue. But ∇R is not strictly positve and therefore it can't be the ground state, and so we can conclude that there is exactly one negative eigenvalue.

Now we extend our result to general linearities f. This is achieved by considering a one-paramter family of equations and interpolating between the result above and with what we are trying to achieve. Specifically, consider

$$\Delta\psi_\tau - \psi_\tau + (1-\tau)\psi_\tau^{2\sigma+1} + \tau f(\psi_\tau^2)\psi_\tau = 0, \tag{19}$$

where $\tau \in [0,1]$. Now, let R_τ be the unique ground state of (19). Also let the linearization of (19) be $L_+(\tau)$. In the previous paragraph it was demonstrated that $L_+(0) = L_+$ has exactly one negative eigenvalue. Before we go on, we need to make an assumption about the dimension of the null space of L_+. We assume that all zero eigenvalues are generated by the translation invariance of equation (2), which means that the null space of L_+ has dimension equal to the dimension of the domain we are in. To continue, we have that the null space of $L_+(\tau)$ has dimension N for any τ. Furthermore, we know that it is spanned by $\frac{\partial R_\tau}{\partial x_i}$ for $i = 1, ..., N$. Now, for any i, $\frac{\partial R_\tau}{\partial x_i}$ is not the ground state, and so $L_+(\tau)$ has atleast one negative eigenvalue. Notice that as one varies $\tau \in [0,1]$, the eigenvalues of $L_+(\tau)$ will vary continuously as well. One can do this until they get to $L_+(1)$, being the linearization with the general non-linearity. If the number of negative eigenvalues were to increase, the multiplicity of the zero eigenvalue would also go up, by the continuity of the parameter τ and the eigenvalues of $L_+(\tau)$. But this contradicts our assumption. Thus we see that $L_+(1)$ has exactly one negative eigenvalue. ■

Now assume that

$$\mathrm{Ker} L_+ = span\{\nabla R\} \;\; and \;\; \mathrm{Ker} L_- = span\{R\}.$$

For more details on this matter we refer the reader to [FGJS], [BL1], and [BL2].

We now need a proposition that allows one to estimate the quantity

$$\inf_{\substack{\|w\|=1 \\ w\perp KerL,R}} \langle w, Lw\rangle.$$

We assume here that for the general class of nonlinear interactions $f(|\psi|^2)$ that given $E > 0$ $\exists!$ ground state $R(E)$, where $R > 0$ and $R \in H^1$.

Proposition 6 *Let R satisfy equation (2), where R is real valued ground state, and so $R = R(E)$. (Here $R(E)$ is the local minimum of H_E for a given E.) Then*

$$\langle R, L_+^{-1}R\rangle < 0.$$

Proof

We now have

$$-\Delta R - f(R^2)R = -ER$$

and we differentiate with respect to E. We get

$$-\Delta\frac{\partial R}{\partial E} - 2f'(R^2)R^2\frac{\partial R}{\partial E} - f(R^2)\frac{\partial R}{\partial E} + R + E\frac{\partial R}{\partial E} = 0$$

which gives

$$[-\Delta - 2f'(R^2)R^2 - f(R^2) + E]\frac{\partial R}{\partial E} = -R.$$

So

$$L_+\frac{\partial R}{\partial E} = -R. \tag{20}$$

Also, we can invert L_+ since R is not in its kernal and so we can write

$$\langle R, L_+^{-1}R\rangle = -\int R\frac{\partial R}{\partial E} = -\frac{\partial}{\partial E}\left(\frac{1}{2}\int R^2\right) = -N'(R) < 0. \blacksquare$$

We now use this proposition to demonstrate that if

$$\alpha = \min_h \langle h, L_+h\rangle$$

then we have $\alpha > 0$, given that $\langle h, R\rangle = 0$.

For the specific nonlinearity $f(|\psi|^2) = \psi^{2\sigma}$ we can argue as follows. It is well known that the infimum of the Rayleigh functional (18) over all $u \in H^1$ is attained at R. See [MW] for example. As mentioned previously, after taking the second Frechet derivative of J about R, one finds for functions where $\langle h, R\rangle = 0$ that

$$\langle L_+h, h\rangle + k^2(\sigma d - 2)\langle h, \Delta R\rangle^2 \geq 0.$$

This means for $\sigma \leq 2/d$ (where d is the dimension) we have

$$\langle L_+ h, h \rangle \geq 0.$$

Therefore there exists a minimum for

$$\alpha = \min_h \langle h, L_+ h \rangle, \tag{21}$$

where $\|h\|_2 = 1$ and $\langle h, R \rangle = 0$.

Now, the corresponding condition to $\sigma \leq 2/d$ for the general nonlinearity $f(|\psi|^2)$ is the condition

$$\frac{d}{dE} \|R(E)\|^2 > 0.$$

Assuming this we can continue in the following way. Assuming that α is attained for the function h_*, by the theory of Lagrange multipliers we know that h_* satisfies the Euler-Lagrange equation

$$\partial_{h_*} \left[\frac{1}{2} \langle h, L_+ h \rangle - \frac{\lambda}{2} \|h\|^2 - \beta \langle R, h \rangle \right] = 0 \tag{22}$$

which means that we have

$$L_+ h_* - \lambda h_* - \beta R = 0. \tag{23}$$

Multiplying this by h_* we get

$$\lambda = \langle L_+ h_*, h_* \rangle \tag{24}$$

by noticing that

$$\|h_*\| = 1 \tag{25}$$

and

$$\langle R, h_* \rangle = 0. \tag{26}$$

Recall that L_+ has one negative eigenvalue that we shall denote as λ_0 and its corresponding eigenfunction as h_0. Now, let $\lambda = \lambda_0$ and take the inner product of (23) with h_0 to obtain

$$\langle L_+ h_*, h_0 \rangle - \lambda_0 \langle h_*, h_0 \rangle - \beta \langle R, h_0 \rangle = 0.$$

But this means that

$$\langle h_*, L_+ h_0 \rangle = \lambda_0 \langle h_*, h_0 \rangle + \beta \langle R, h_0 \rangle$$

or

$$\lambda_0 \langle h_*, h_0 \rangle = \lambda_0 \langle h_*, h_0 \rangle + \beta \langle R, h_0 \rangle.$$

This means that either $\beta = 0$ or that $\langle h_*, h_0 \rangle = 0$. This is not true however. Further, if $\beta = 0$ then α is an eigenvalue of L_+, which one can see by considering equations (24) and (23). One gets

$$L_+ h_* = \alpha h_* + \beta R.$$

But we know that L_+ has only one negative eigenvalue which implies that $h_* > 0$. This would contradict equation (26) however.

Consider $\lambda \in (\lambda_0, 0]$ and the function

$$g(\lambda) = \langle (L_+ - \lambda)^{-1} R, R \rangle. \tag{27}$$

In this way we have

$$h_* = \beta (L_+ - \alpha)^{-1} R$$

so equation (26) is the same as

$$\beta \langle (L_+ - \alpha)^{-1} R, R \rangle = 0$$

or

$$g(\alpha) = 0. \tag{28}$$

We claim that α cannot be negative. Notice that

$$g'(\lambda) = \|(L_+ - \lambda)^{-1} R\|^2, \tag{29}$$

which implies that g is increasing on $(\lambda_0, 0]$. To guarantee that there is no zero of g on $(\lambda_0, 0]$, all we need is

$$g(0) = \langle L_+^{-1} R, R \rangle \leq 0. \tag{30}$$

But this is true so long as

$$\frac{d}{dE} \|R(E)\|^2 > 0. \tag{31}$$

Again, recall that this condition reduces to $\sigma < 2/d$ for the nonlinearity $f(\psi) = |\psi|^{2\sigma}$. Thus, we have that $\alpha \geq 0$ as desired, as long as we have condition (31).

What remains to look at is the operator L_-. This is much easier for it has no negative eigenvalues. To make the argument as precise as possible consider taking as our metric d the function

$$d^2(\phi(t), M_R) = \inf \left\{ \|\nabla \phi(\cdot + x_0, t) e^{i\gamma} - \nabla R\|_{L^2}^2 + E \|\phi(\cdot + x_0, t) e^{i\gamma} - R\|_{L^2}^2 \right\}. \tag{32}$$

Minimization of

$$\|\nabla \phi(\cdot + x_0, t) e^{i\gamma} - \nabla R\|_{L^2}^2 + E \|\phi(\cdot + x_0, t) e^{i\gamma} - R\|_{L^2}^2$$

over x_0 and γ gives

$$\int f(R^2)Rv dx = 0. \tag{33}$$

Now consider

$$\inf \frac{\langle L_- v, v\rangle}{\langle v, v\rangle} \tag{34}$$

subject to (31). If we assume that (32) is identical to zero, then it is attained at R. But this would contradict (31), and so we can conclude that it is positive or zero. Thus, we are not quite finished.

In order to conclude the proof of Lemma 2, we outline the idea of the remaining steps. In addition to the assumption that the L^2 norm of the perturbed soliton ψ is the same as that of the ground state R, and the assumption given by (31), we require that

$$\int [f(R^2) + 2R^2 f'(R^2)|\partial_{x_i} R|^2] dx \neq 0 \quad \text{for} \quad i = 1, 2, ...d. \tag{35}$$

Notice that we obtained that $\alpha \geq 0$. Consider the metric given by d^2 as defined previously. In [MW] it was demonstrated that if $x_0 = x_0(t)$ and $\gamma = \gamma(t)$ are chosen so as to minimize this metric then one also obtains

$$\int [f(R^2) + 2R^2 f'(R^2)] \frac{\partial R}{\partial x_j} u dx = 0 \quad \text{for} \quad j = 1, ..., d$$

where u is the real part of the pertubation of the ground state R. This together with (35), guarantees that α is strictly positive.

Now, take note of the assumption

$$\int |\psi|^2 = \int R^2,$$

where $\psi = R + w$, and $w = u + iv$. This gives

$$\langle R + w, \overline{R + w}\rangle = \langle R, R\rangle$$

$$\implies \langle u + iv, R\rangle + \langle R, u - iv\rangle + \langle u + iv, u - iv\rangle = 0$$
$$\implies 2\langle u, R\rangle = -\frac{1}{2}[\langle u, u\rangle + \langle v, v\rangle].$$

Following [MW], one takes $u = u_{||} + u_{\perp}$ and using the above one rewrites

$$\langle L_+ u, u\rangle = \langle L_+ u_{||}, u_{||}\rangle + 2\langle L_+ u_{||}, u_{\perp}\rangle + \langle L_+ u_{\perp}, u_{\perp}\rangle$$

as

$$\langle L + u, u\rangle \geq \frac{1}{4}\langle L_+ R, R\rangle[\langle u, u\rangle + \langle v, v\rangle]^2 - c'\|w\|^2_{H^1}\|\nabla w\|_{H^1} + c''\{\langle u, u\rangle - \frac{1}{4}[\langle u, u\rangle + \langle v, v\rangle]\}^2.$$

Similarly, one can find that

$$\langle L_- v, v\rangle \geq c'''\|v\|^2_{H^1}.$$

Lemma 2 follows from this. ■

III. PINNING OF SOLITONS FOR THE NLS WITH POTENTIAL

In this section we consider the stationary non-linear Schrodinger equation given by

$$(-\Delta + V_\varepsilon + \lambda)\psi - f(\psi) = 0 \tag{36}$$

where $V_\varepsilon(x) = V(\varepsilon x)$ is an external potential and $\psi \in H^2$. Here $\varepsilon \in \mathbb{R}$ is a small paramter.

Consider the situation when we take our non-linearity to be $f(\psi) := g(|\psi|^2)$. To ensure that we have positive, spherically symmetric solitons as solutions to (36) we require the following conditions on the nonlinearity $g(|\psi|^2)$:

$$0 \leq \lim_{s \to 0} g(|s|) < \omega \in \mathbb{R}^+ \text{ and}$$

$$0 \leq \lim_{s \to \infty} g(s) \leq Cs^p,$$

$$\text{where for } d > 2 \text{ we have } 1 < p < 1 + \frac{4}{d-2}.$$

When $d = 1$ or 2 then $p \in (1, \infty)$. And finally

$$\exists c > 0 \text{ such that } \int_0^a sg(s^2) > \frac{1}{2}\omega c^2.$$

More details of this nature can be found in [BL1] and [BL2].

Through out this section however we consider the more general non-linearity $f(\psi)$ and so we need the following conditions on f to ensure that we again have positive spherically symmetric solitons. Note that we take $\Omega \subset \mathbb{R}^d$ and that $H_0^1(\Omega)$ represents the closure in $H^1(\Omega)$ of C^∞ functions $\Omega \mapsto \mathbb{R}$ which are compactly supported in Ω. We require:

- $f \in C(H^1(\mathbb{R}^d), H^{-1}(\mathbb{R}^d))$

and $\exists$ $\{f_1, ..., f_k\} \in C(H^1(\mathbb{R}^d), H^{-1}(\mathbb{R}^d))$ such that $f = f_1 + \cdots + f_k$ where each f_i satisfies

- $F_i \in C^1(H_0^1(\Omega), \mathbb{R})$ such that $F_i' = f_i$.

  For the following we require that $\exists r, p \in [2, \frac{2d}{d-2})$ such that

  $$f : H_0^1(\Omega) \longrightarrow L^p(\Omega) \hookrightarrow H^{-1}(\Omega).$$

  If $d = 1$ then we require that $r, p \in [2, \infty]$ and if $d = 2$ then we have $r, p \in [2, \infty)$.

- for every $M > 0$ $\exists$ $C(M) < \infty$ such that

  $$\|f(v) - f(u)\|_{L^p} \leq C(M)\|v - u\|_{L^r}$$

for every $u, v \in H_0^1(\Omega)$ such that

$$\|u\|_{H^1} + \|v\|_{H^1} \leq M.$$

Also $Im(f(u)\bar{u}) = 0$ a.e. on Ω for every $u \in H_0^1(\Omega)$. And finally

- the Energy:

$$E(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - F(u)$$

$\forall u \in H_0^1(\Omega)$. We take that $E \in C^1(H_0^1(\Omega), \mathbb{R})$.

We refer the reader to [Caze]. Before going on we would like to mention that the upon the assumptions of f above, we can have the following generalized non-linearity:

$$f(\psi)(x) = h(x, \psi(x)) + (W * |\psi|^2)(x)\psi,$$

where $W : \mathbb{R}^d \mapsto \mathbb{R}$ is an even function where $W \in L^q(\Omega) + L^\infty(\Omega)$, for $q \geq 1, q > n/4$. Further

$$h : \mathbb{R}^d \times [0, \infty) \mapsto \mathbb{R}$$

is continuous and measurable in ψ and $h(x, 0) = 0$ a.e. on $\mathbb{R}^d$. Also, if $d \geq 2$, then we assume $\exists$ constants C and $\alpha \in [0, \frac{4}{n-2})$ so that

$$|h(x, v) - h(x, u)| \leq C(1 + |u|^\alpha + |v|^\alpha)|v - u|,$$

for almost all $x \in \mathbb{R}^d$ and all $u, v \in \mathbb{R}$. If $d = 2$ then let $\alpha \in [0, \infty)$. If on the otherhand $d = 1$ then assume that for every M that there is an $L(M)$ such that

$$|h(x, v) - h(x, u)| \leq L(M)|v - u|$$

for almost all $x \in \mathbb{R}$ and all $u, v \in \mathbb{R}$ where $|u| + |v| \leq M$. All of this guarantees global well posedness of our problem. One additional assumption before we continue:

$$f'(\psi) \in L^\infty.$$

This allows one to obtain a positive bound that we will require later. Again, we refer the reader interested in more details to [Caze] as well as to [FGJS].

The main focus of this section is to examine soliton like solutions near critical points of $V_\varepsilon(x)$ for equation (36). For simplicity, and with no loss of generality, let us assume that zero is a critical value of V at x_0, i.e.

$$V(x_0) = 0 \ \ \& \ \ \nabla V(x_0) = 0. \tag{37}$$

From Section 1, we already know that (36) with $\varepsilon = 0$ has soliton type solutions, that we shall call R_0. We have $R_0 > 0$ and

$$(\Delta + \lambda)R_0 - f(R_0) = 0, \tag{38}$$

where R_0 is real valued. Notice that R_0 is a solution to (38) and so we have that $R_q(x) = R_0(x+q)$ is also a solution to (38) $\forall q \in \mathbb{R}$, due to translation invariance.

Our goal for this section is to outline a proof for the following.

Theorem 7 *For ε sufficiently small, $\exists$ a solution to* (36) *of the form*

$$R_\varepsilon(x) = R_0(x - a(\varepsilon)) + \xi$$

where $\xi = O(\varepsilon^2)$ and $a(\varepsilon) = x_0 + O(\varepsilon)$.

Idea of Proof :

Take

$$F(\psi, \varepsilon) = (-\Delta + V_\varepsilon + \lambda)\psi - f(\psi)$$

so

$$F : H^2 \times \mathbb{R} \longrightarrow L^2$$

where ψ is real valued.

We wish to solve the equation $F(\psi, \varepsilon) = 0$. Notice that we know that $F(R_0, 0) = 0$ but that the Implicit Function Theorem cannot be applied directly since

$$\partial_\psi F(R_0, 0) = -\Delta + \lambda - f'(R_0)$$

is not invertible. To see this notice that

$$\frac{\partial}{\partial q}\left[(-\Delta + \lambda)R_q - f(R_q)\right]_{q=0} = 0 \tag{39}$$

and so

$$(-\Delta + \lambda)\nabla R_0 - f'(R_0)\nabla R_0 = 0$$

$$\Longrightarrow \;\; (-\Delta + \lambda - f'(R_0))\nabla R_0 = 0$$

or

$$\partial_\psi F(R_0, 0)\nabla R_0 = 0. \tag{40}$$

We have to consider

$$L := \partial_\psi F(R_0, 0)$$

. Thus,

$$\text{Null } L = \text{span}\left\{\frac{\partial R_0}{\partial x_j}, iR_0\right\}, \tag{41}$$

where $j = 1, \ldots, n$. In [MW2], Weinstein had this result for dimensions one and three. He also had it in general, except that he was missing the fact that any solution of

$$\left(-\frac{d^2}{dr^2} - \frac{n-1}{r}\frac{d}{dr} + 1 - pu_0^{p-1}\right) f = 0$$

where

$$f'(0) = 0 \quad \text{and} \quad f(0) = 1$$

is unbounded. This however was established in [KW].As an improvement McLeod in [KM] provided a similar but more powerful result where the last term is replaced by a more general function whose properties are the same as our nonlinearity. For more details on (41) see [DMAS] as well. Equation (41) is claimed to be true in Appendix D of [FGJS].

We have $L := \partial_\psi F(R_0, 0)$ and let $N^\perp := [Null\ \partial_\psi F(R_0, 0)]^\perp$ be the restriction of L to the space orthogonal to (41). We also define P_q to be the projection onto Null $L = N$, and so we also have $\bar{P}_q = \mathbb{I} - P_q$, which is just the orthogonal projection onto $N^\perp$. Having said this, we have that for ψ in a closed neighborhood of

$$M = \{R_q \big| q \in \mathbb{R}\}$$

we express ψ as

$$\psi = R_q + u \tag{42}$$

so that

$$\langle \nabla R_q, u \rangle = 0$$

i.e. $u \in Ran\bar{P}_q$. This expression is permissable because ψ is an element of a Hilbert space and so within any closed subspace, it can be decomposed in this way.

Now define

$$\phi(u, q, \varepsilon) := F(R_q + u, \varepsilon)$$

by definition of ψ. Also notice that $\phi(0, q, 0) = F(R_q, 0) = 0$. We then decompose ϕ to obtain

$$P_q \phi(u, q, \varepsilon) = 0 \tag{43}$$

and

$$\bar{P}_q\phi(u,q,\varepsilon) = 0. \tag{44}$$

In this way we have two equations with two unknowns.

Before we continue, we will demonstrate a theorem where we estimate the function u. It requires the use of a lemma which ensures that our operator $\bar{L}_{q,\varepsilon}$ is bounded below by a positive constant C_L. We demonstrate this first.

Lemma 8

$$\|\bar{L}_{q,\varepsilon}\upsilon\|_{L^2} \geq C_L\|\upsilon\|_{H^2} \tag{45}$$

where $\upsilon \in H^2 \cap N^{\perp}$.

Idea of Proof : First off, note that

$$\bar{L}_{q,\varepsilon} = \bar{P}_q[-\Delta + V_\varepsilon + \lambda - f'(R_q)] = \bar{L}_{q,0} + V_\varepsilon,$$

where

$$\bar{L}_{q,0} = \bar{P}_q[-\Delta + \lambda - f'(R_q)].$$

We begin by noticing that since zero is not an eigenvalue of our operator and $\bar{L}^*_{q,0} = \bar{L}_{q,0}$, we have that

$$\|\bar{L}_{q,0}\upsilon\|_{L^2} \geq \operatorname{dist}(0, \sigma(\bar{L}_{q,0}))\|\upsilon\|_{L^2},$$

where $\sigma(\bar{L}_{q,0})$ is the spectrum of the operator in question.

Now, for simplicity during the proof, let $\bar{L}_{q,\varepsilon} := L$. Thus we know $\exists\gamma \in \mathbb{R}^+$ such that

$$\|L\upsilon\|_{L^2} \geq \gamma\|\upsilon\|_{L^2}.$$

Rewriting L as $(1-\delta)L + \delta L$ and noticing that $\langle\upsilon, L\upsilon\rangle \leq \|\upsilon\|_{H^1}\|L\upsilon\|_{L^2}$ we can write

$$\begin{aligned}\langle\upsilon, L\upsilon\rangle &\geq (1-\delta)\gamma\|\upsilon\|^2_{L^2} + \delta\langle\upsilon, L\upsilon\rangle\\ &= (1-\delta)\gamma\|\upsilon\|^2_{L^2} + \delta\langle\upsilon, (-\Delta + V_\varepsilon + \lambda - f'(R_q))\upsilon\rangle\end{aligned}$$

where it is understood that the operator is being projected away from its kernal. We then see that

$$\begin{aligned}\|L\upsilon\|_{L^2}\|\upsilon\|_{H^1} &\geq (1-\delta)\gamma\|\upsilon\|^2_{L^2} + \delta\langle\upsilon, -\Delta\upsilon\rangle + \delta\langle\upsilon, V_\varepsilon\upsilon\rangle + \delta\lambda\langle\upsilon,\upsilon\rangle - \delta\langle\upsilon, f'(R_q)\upsilon\rangle\\ &\geq (1-\delta)\gamma\|\upsilon\|^2_{L^2} + \delta\|\nabla\upsilon\|^2_{L^2} + \delta C_V\|\upsilon\|^2_{L^2} + \delta\lambda\|\upsilon\|^2_{L^2} - \delta\|f'(R_q)\|_{L^\infty}\|\upsilon\|^2_{L^2}\\ &\geq \min\left(\delta, (1-\delta)\gamma + \delta C_V + \delta\lambda - \delta\|f'(R_q)\|_{L^\infty}\right)\|\upsilon\|^2_{H^1}\end{aligned}$$

which means we have

$$\|Lv\|_{L^2} \geq \omega \|v\|_{H^1}.$$

via Holder. We are not done yet.

Using a similar trick, we take

$$\begin{aligned}
\|Lv\|_{L^2} &= (1+\delta')\|Lv\|_{L^2} - \delta'\|Lv\|_{L^2} \\
&\geq (1+\delta')\omega\|v\|_{H^1} - \delta'\|(-\Delta + \lambda + V_\varepsilon - f'(R_q))v\|_{L^2} \\
&\geq (1+\delta')\omega\|v\|_{H^1} - \delta'\left[\|(\lambda + V_\varepsilon)v\|_{L^2} + \delta'\|\Delta v\|_{L^2} - \|f'(R_q)v\|_{L^2}\right] \\
&\geq (1+\delta')\omega\|v\|_{H^1} - \delta'(\lambda + C_V)\|v\|_{H^1} + \delta' M_{f'} + \delta'\|\Delta v\|_{L^2} \\
&\geq \min\{(1+\delta')\omega - \delta'(\lambda + C_V), \delta')\|v\|_{H^2}.
\end{aligned}$$

Letting C_L be that minimum and noting that $M_{f'} = \|f'(R_q)v\|_{L^2} \leq \|f'(R_q)\|_{L^\infty}\|v\|_{L^2} < \infty$ we have our claim. We need to mention only one more thing. We want $C_L > 0$. This means that for $\delta' > 0$ we have

$$C_V < \frac{1+\delta'}{\delta'}\omega - \lambda.$$

This is a constraint on the potential V_ε. ■

We have completed the proof of the lemma and now wish to use this result to prove

Lemma 9 *The solution* $u(q, \varepsilon)$ *of* (44) *satisfies*

$$u = O(\varepsilon^2).$$

Idea of Proof : We begin by expanding (44) in u remembering that

$$\phi(u, q, \varepsilon) = F(R_q + u, \varepsilon) = 0.$$

Let

$$\bar{L}_{q,\varepsilon} = \bar{P}_q \partial_\psi F(R_q, \varepsilon).$$

Then we have

$$\bar{P}_q F(R_q, \varepsilon) + \bar{L}_{q,\varepsilon} u + \bar{P}_q N(u, \varepsilon) = 0. \tag{46}$$

We already know that $\bar{L}_{q,0}$ is invertible. But notice that

$$\begin{aligned}
\bar{L}_{q,\varepsilon} &= \bar{P}_q\left[-\Delta + V_\varepsilon + \lambda - f'(R_q)\right] &(47)\\
&= \bar{L}_{q,0} + V_\varepsilon = \bar{L}_{q,0} + O(\varepsilon), &(48)
\end{aligned}$$

so that $\bar{L}_{q,\varepsilon}$ is also invertible. From (46) we find

$$u = -\bar{L}_{q,\varepsilon}^{-1}\, \bar{P}_q\, [F(R_q, \varepsilon) + N(u, \varepsilon)].$$

But

$$F(R_q, \varepsilon) = (-\Delta + \lambda)R_q - f(R_q) + V_\varepsilon R_q = V_\varepsilon R_q,$$

so

$$u = -\bar{L}_{q,\varepsilon}^{-1}\, \bar{P}_q\, [V_\varepsilon R_q + N(u, \varepsilon)]. \tag{49}$$

We know that $\bar{L}_{q,\varepsilon}^{-1}$ is bounded above.

We begin with an argument for the specific non-linearity $f(\psi) = |\psi|^2\psi$, and then make some comments about the general non-linearity. For the specific non-linearity

$$N(u, \varepsilon) = 3R_q u^2 + \frac{u^3}{2}.$$

Notice that $\bar{L}_{q,\varepsilon}^{-1} : L^2 \mapsto H^2$. Now, since $\psi \in H^2$,and $\psi = R_q + u$, we know that u and R_q both live in H^2. It is easy to verify the Sobolev imbedding

$$H^2 \hookrightarrow L^6. \tag{50}$$

From this we can arrive at

Proposition 10 *R_q and u are in L^6. Further, $u^3 \in L^2$ and $u^2 R_q \in L^2$.*

Proof : The first part of the proposition follows immediately from the Sobolev Imbedding just mentioned. Now, we have that $u \in L^6$ which means that we have

$$\int (u^3)^2 < \infty$$

which means that $u^3 \in L^2$. To see the last claim notice that

$$\|u^2 R_q\|_{L^2}^2 \le \|u^4\|_{L^1} \|R_q^2\|_{L^\infty}.$$

We know that $\|R_q^2\|_{L^\infty} < \infty$ and

$$\|u^4\|_{L^1} = \|u\|_{L^4}^4.$$

We have that $u \in L^6$ and $u \in L^2$, (since it is an H^2 function) which means that $u \in L^6 \cap L^2$ and so $u \in L^4$. Thus, our claim is true. ■

Now using Proposition 2 and Lemma 3 we will estimate the H^2 norm of u. We have

$$\begin{aligned} \|u\|_{H^2} &\le 3\|\bar{L}_{q,\varepsilon}^{-1} R_q u^2\|_{H^2} + \|\bar{L}_{q,\varepsilon}^{-1} u^3\|_{H^2} + \|\bar{L}_{q,\varepsilon}^{-1} V_\varepsilon R_q\|_{H^2} \\ &\le C_L^{-1} \left\{3\|R_q u^2\|_{L^2} + \|u^3\|_{L^2} + \|V_\varepsilon R_q\|_{L^2}\right\} \\ &\le C_L^{-1} \left\{3\|R_q^2\|_{L^\infty}^{\frac{1}{2}} \|u\|_{L^4}^2 + \|u\|_{L^6}^3 + \|V_\varepsilon^2\|_{L^\infty} \|R_q^2\|_{L^2}\right\} \end{aligned}$$

Recall that the IFT was used to demonstrate the existence of u, such that $u = u(q, \varepsilon)$ and $u(q, 0) = 0$. By the regularity of u and the aforementioned property of u, it is reasonable to assume that $\|u\|_{H^2} < \varepsilon$. Then from the fact that $u \in L^2 \cap L^6$ and the imbedding (50), we have

$$\|u\|_{L^4} \leq \|u\|_{H^2} < \varepsilon \quad \text{and} \quad \|u\|_{L^6} \leq \|u\|_{H^2}.$$

Thus, our H^2 estimate for u yields

$$\|u\|_{H^2} \leq C_L^{-1} \left\{3\|R_q^2\|_{L^\infty}\varepsilon^4 + \varepsilon^3 + \|R_q^2\|_{L^2}\varepsilon^2\right\}$$

where both norms of R_q^2 are finite and where we took $\|V_\varepsilon^2\|_{L^\infty} \leq \varepsilon^2$. This means that we have

$$u = O(\varepsilon^2).$$

This completes the estimate of u in terms of ε.

To extend this estimate to the general non-linearity $f(\psi)$, we first need to notice that

$$N(u, \varepsilon) = \frac{f''(R_q)u^2}{2} + \frac{f'''(R_q)u^3}{3!},$$

for the general $f(\psi)$. The last line of our estimate for u will now look like

$$\|u\|_{H^2} \leq C_L^{-1} \left\{\|(f'')^2\|_{L^\infty}^{\frac{1}{2}}\|u\|_{L^4}^2 + \|(f''')^2\|_{L^\infty}^{\frac{1}{2}}\|u\|_{L^6}^3\right\}.$$

The proof will work identically to the specific case if in addition we assume that

$$\|(f^{(j)})^2\|_{L^\infty} < \infty \quad \text{for} \quad j = 2, 3.$$

■

We now return to where we left off. Recall that we had two equations, (43) and (44), in two unknowns. We will attempt to solve (44) first, for u. To simplify, let

$$G(u, q, \varepsilon) = \bar{P}_q\phi(u, q, \varepsilon).$$

We have that

$$G : Ran\bar{P}_q \times \mathbb{R}^n \times \mathbb{R} \longrightarrow Ran\bar{P}_q$$

is C^1 in u, and that

$$G(0, q, 0) = \bar{P}_q[(-\Delta + \lambda)R_q - f(R_q)] = 0 \ \ \forall q \in \mathbb{R}^n,$$

but

$$\partial_u G(0, q, 0) = \partial_u \bar{P}_q F(R_q, 0)$$

is invertible, being because we projected away from the zero eigenspace. The Implicit Function Theorem is therefore applicable and so we have that $\exists$ a solution $u = u(q, \varepsilon)$ to (44) such that $u(q, 0) = 0$. We take this u and plug it into (43) to obtain

$$P_q \phi(u(q, \varepsilon), q, \varepsilon) = 0, \tag{51}$$

our reduced equation. Since our projection P_q is given by

$$P_q = \sum_i \frac{|\partial_{x_i} R_q\rangle\langle \partial_{x_i} R_q|}{\|\partial_{x_i} R_q\|_{L^2}},$$

we can rewrite this equation as

$$\sum_i \big\langle \partial_{x_i} R_q, \phi(u(q, \varepsilon), q, \varepsilon)\big\rangle = 0. \tag{52}$$

Notice that

$$P_q \phi(u, q, \varepsilon) = P_q F(R_q + u, \varepsilon)$$

$$\implies \langle \partial_{x_i} R_q, [(-\Delta + \lambda)R_q - f(R_q)]\rangle = 0 \;\; \forall q \in \mathbb{R}.$$

We want to solve (51) for q. Notice however

$$\partial_q \big\langle \partial_{x_i} R_q, \phi(u(q, 0), q, 0)\big\rangle = \langle \partial_{x_i} R_q, [\partial_u \phi(0, q, 0)\partial_q u(q, 0) + \partial_q \phi(0, q, 0)]\rangle = 0$$

and so we cannot use the Implicit Function Theorem.

Proposition 11 $\forall \varepsilon \leq \varepsilon_0 \;\; \exists \, q_0$ *which solves*

$$P_{q_0} \phi(u(q_0, \varepsilon), q_0, \varepsilon) = 0.$$

Proof : Notice that we have

$$P_q \phi = \partial_x R_q \langle \partial_x R_q, \phi\rangle. \tag{53}$$

Denote

$$s(q, \varepsilon) := \langle \partial_x R_q, \phi\rangle.$$

We wish to solve

$$s(q_0, \varepsilon) = 0 \tag{54}$$

for q_0. The next lemma provides an explicit formula for $s(q, \varepsilon)$ which allows us to solve (54). The proof of the following lemma depends on three important facts that allows one to make the necessary cancellations which gives the required equation. These are claims (55),(56), and (57).

Lemma 12

$$s(q,\varepsilon)=\frac{\varepsilon^2 q}{2}V''(0)\int\left(\frac{\partial R_0}{\partial x}\right)^2 x^2-\frac{\varepsilon^3}{4}V'''(0)\int R_0^2(x)x^2+O(\varepsilon^2q^2)+O(\varepsilon^3 q).$$

Proof : We expand $s(q,\varepsilon)$ in ε about 0. One would get

$$\partial_\varepsilon s(q,0)=\sum_i\langle\partial_{x_i}R_q,[\partial_u\phi(0,q,0)\partial_\varepsilon u(q,0)+\partial_\varepsilon\phi(0,q,0)]\rangle.$$

The second term gives

$$\begin{aligned}\partial_\varepsilon^2 s(q,0)=\sum_i\big\langle\partial_{x_i}R_q,[\{\partial_u^2\phi(u,q,\varepsilon)\partial_\varepsilon u(q,\varepsilon)+\partial_\varepsilon\partial_u\phi(u,q,\varepsilon)\}\partial_\varepsilon u(q,\varepsilon)\\ +\{\partial_u\phi(u(q,\varepsilon),q,\varepsilon)\partial_\varepsilon^2u(q,\varepsilon)\}+\partial_\varepsilon^2\phi(u,q,\varepsilon)]\big\rangle\Big|_{\substack{u=0\\ \varepsilon=0}}\end{aligned}$$

Finally, the third term gives

$$\begin{aligned}\partial_\varepsilon^3 s(q,0)=\sum_i\Big\langle\partial_{x_i}R_q,\Big[\Big\{\frac{\partial^3\phi}{\partial u^3}(0,q,0)\frac{\partial u}{\partial\varepsilon}(q,0)+\partial_\varepsilon\frac{\partial^2\phi}{\partial u^2}(0,q,0)\Big\}\Big(\frac{\partial u}{\partial\varepsilon}(q,0)\Big)^2\\ +2\Big\{\frac{\partial^2\phi}{\partial u^2}(0,q,0)\frac{\partial u}{\partial\varepsilon}(q,0)\frac{\partial^2 u}{\partial\varepsilon^2}(q,0)\Big\}+\partial_\varepsilon^2\frac{\partial\phi}{\partial u}(0,q,0)\frac{\partial u}{\partial\varepsilon}(q,0)+\partial_\varepsilon\frac{\partial\phi}{\partial u}(0,q,0)\frac{\partial^2 u}{\partial\varepsilon^2}(q,0)\\ +\Big(\frac{\partial^2\phi}{\partial u^2}(0,q,0)\frac{\partial u}{\partial\varepsilon}(q,0)+\partial_\varepsilon\frac{\partial\phi}{\partial u}(0,q,0)\Big)\frac{\partial^2 u}{\partial\varepsilon^2}(q,0)+\frac{\partial\phi}{\partial u}(0,q,0)\frac{\partial^3 u}{\partial\varepsilon^3}(q,0)+\partial_\varepsilon^3\phi(0,q,0)\Big]\Big\rangle.\end{aligned}$$

We claim

Three Important Facts

$$P_q\partial_u\phi(0,q,0)=0, \tag{55}$$

$$\frac{\partial u}{\partial\varepsilon}(q,\varepsilon)\Big|_{\varepsilon=0}=0, \tag{56}$$

and

$$\partial_\varepsilon\frac{\partial\phi}{\partial u}(0,q,0)=0. \tag{57}$$

To see (55) is easy because note that

$$\partial_u\phi(0,q,0)=-\Delta+\lambda-f'(R_q)$$

and that P_q projects this operator onto its kernel. Thus the claim is true.

To see (56) we notice that we have

$$\partial_\varepsilon\bar{P}_qF(R_q+u,\varepsilon)=0$$

$$\Longrightarrow \bar{P}_q \left[-\Delta \partial_\varepsilon u + V'(\varepsilon x) x \psi + V(\varepsilon x) \partial_\varepsilon u + \lambda \partial_\varepsilon u - f'(\psi) \partial_\varepsilon u \right] = 0$$

so at $\varepsilon = 0$ we have $V'(0) = 0$ and

$$\bar{P}_q \left[-\Delta + \lambda - f'(R_q) \right] \frac{\partial u}{\partial \varepsilon}\Big|_{\varepsilon=0} = 0.$$

But notice that $\bar{P}_q \left[-\Delta + \lambda - f'(R_q) \right]$ is invertible and so we conclude that

$$\frac{\partial u}{\partial \varepsilon}(q, \varepsilon)\Big|_{\varepsilon=0} = 0.$$

Finally, to see (57) just notice that

$$\partial_\varepsilon \frac{\partial \phi}{\partial u}(u, q, \varepsilon) = V'(\varepsilon x) x - f''(R_q + u) \frac{\partial u}{\partial \varepsilon}(q, \varepsilon)$$

$$\Longrightarrow \partial_\varepsilon \frac{\partial \phi}{\partial u}(0, q, 0) = -f''(R_q) \frac{\partial u}{\partial \varepsilon}(q, \varepsilon)\Big|_{\varepsilon=0} = 0$$

by (56).

From the above we gather that

$$\partial_\varepsilon^n z(q, 0) = P_q \partial_\varepsilon^n \phi(0, q, 0) \quad n = 1, 2, 3.$$

Recall however that

$$\phi(u, q, \varepsilon) = F(R_q + u, \varepsilon) = F(R_q + u, 0) + V_\varepsilon(x)(R_q + u)$$

$$\Longrightarrow \partial_\varepsilon^n z(q, 0) = P_q V^{(n)}(0) x^n R_q.$$

This means that we now can write

$$s(q, \varepsilon) = \frac{1}{2} V''(0) \left\langle \frac{\partial R_q}{\partial x}, x^2 R_q \right\rangle \varepsilon^2 + \frac{1}{6} V'''(0) \left\langle \frac{\partial R_q}{\partial x}, x^3 R_q \right\rangle \varepsilon^3 + O(\varepsilon^4). \tag{58}$$

From the shape of R_0, we know that $x^2 R_0$ is even and $\frac{\partial R_0}{\partial x}$ is odd

$$\Longrightarrow \quad \left\langle \frac{\partial R_0}{\partial x}, x^2 R_0 \right\rangle = 0.$$

Taylor expand R_q and get

$$R_q(x) = R_0(x) + R_0'(x) q + O(q^2)$$

and notice that

$$\frac{\partial R_q}{\partial x}(x) = \frac{\partial R_0}{\partial x}(x) + O(q).$$

This implies that we have

$$\left\langle \frac{\partial R_q}{\partial x}, x^2 R_q \right\rangle = \left\langle \frac{\partial R_0}{\partial x}, x^2 R_0 \right\rangle + \left\langle \frac{\partial R_0}{\partial x}, x^2 \frac{\partial R_0}{\partial x} \right\rangle q + O(q^2). \tag{59}$$

We also have

$$\begin{aligned}\left\langle \frac{\partial R_q}{\partial x}, x^3 R_q \right\rangle &= \int \frac{\partial R_0}{\partial x} x^3 R_0 + O(q) & (60)\\ &= \frac{1}{2}\int \left(\partial_x R_0^2\right) x^3 + O(q) & (61)\\ &= -\frac{3}{2}\int R_0^2 x^2 + O(q) & (62)\end{aligned}$$

by integrating by-parts. Using (59) and (62) in conjunction with (58) gives the desired result. ■

From here we see that there exists a solution to (54) given by

$$q_0(\varepsilon) = \frac{1}{2}\frac{V'''(0)}{V''(0)}\frac{\int R_0^2(x)x^2}{\int \left(\frac{\partial R_0}{\partial x}\right)^2 x^2}\,\varepsilon + O(\varepsilon^2). \tag{63}$$

Thus, we have the proposition as desired. ■

Now, recall from Theorem 3 that we claimed that there was a solution to (36) of the form

$$R_\varepsilon = R_0(x - a(\varepsilon)) + O(\varepsilon^2).$$

If we let $R_\varepsilon = \psi$, and $a(\varepsilon) = -q(\varepsilon)$, then from equation (42) we get

$$R_\varepsilon = R_0(x - a(\varepsilon)) + u.$$

But from Theorem 4 we know that u is $O(\varepsilon^2)$, and so we are done.

IV. SOME NUMERICAL EFFORTS TOWARDS THE NLS WITH POTENTIAL

A spectral method was used to simulate the dynamics of solitons in an external potential. For periodic problems the spectral method has been shown to be extremely stable and not too computationally expensive, i.e. the time stepping does not have to be absurdly small.

The specific methods used to generate the dynamics is based on a splitting algorithm introduced to me by Mary Pugh and Robert Almgren. It works as follows. Consider the cubic NLS with potential V given by

$$\partial_t \psi - i\Delta\psi - i|\psi|^2\psi + iV(x)\psi = 0 \tag{64}$$

with initial condition

$$\psi(0, x) = \psi_0.$$

One takes (64) and splits it into

$$\partial_t \psi = i\Delta\psi \tag{65}$$

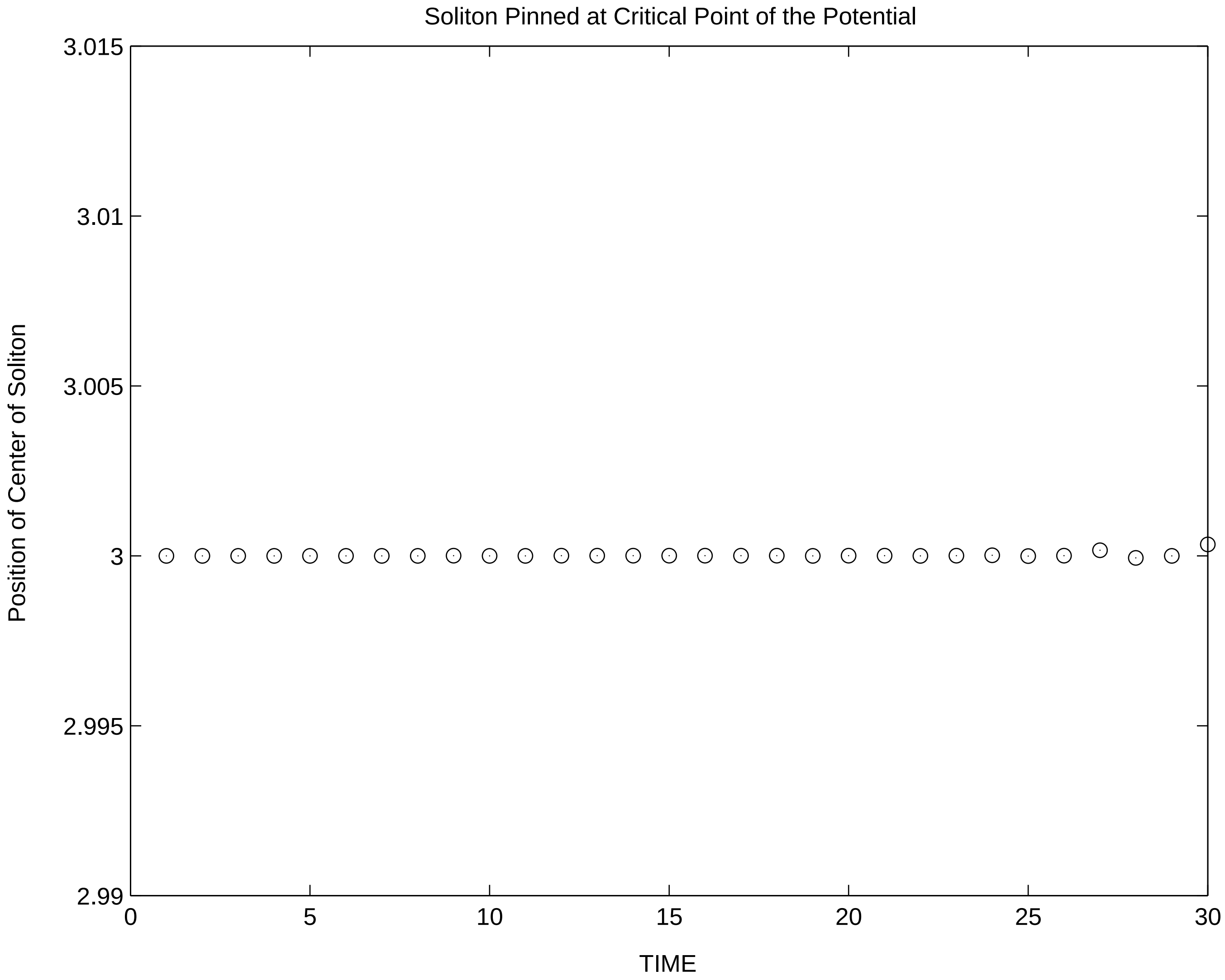

FIG. 2:

and

$$\partial_t \psi = i|\psi|^2\psi - V(x)\psi. \tag{66}$$

One then uses a Fast Forward Transform(FFT) and its inverse(IFFT) to compute derivatives of these functions in accordance with the equation (64).

To decrease the error in the program a Richardson Extrapolation was introduced as well as Strang Splitting. Convergence tests demonstrated that the program was running with an error of dt^4 locally. The Richardson Extrapolation effected the mass conservation, but the error was small.

The potentials we experimented with for the simulations were gaussians i.e. of the form

$$V(x) = \pm\beta exp(-\gamma x^2)$$

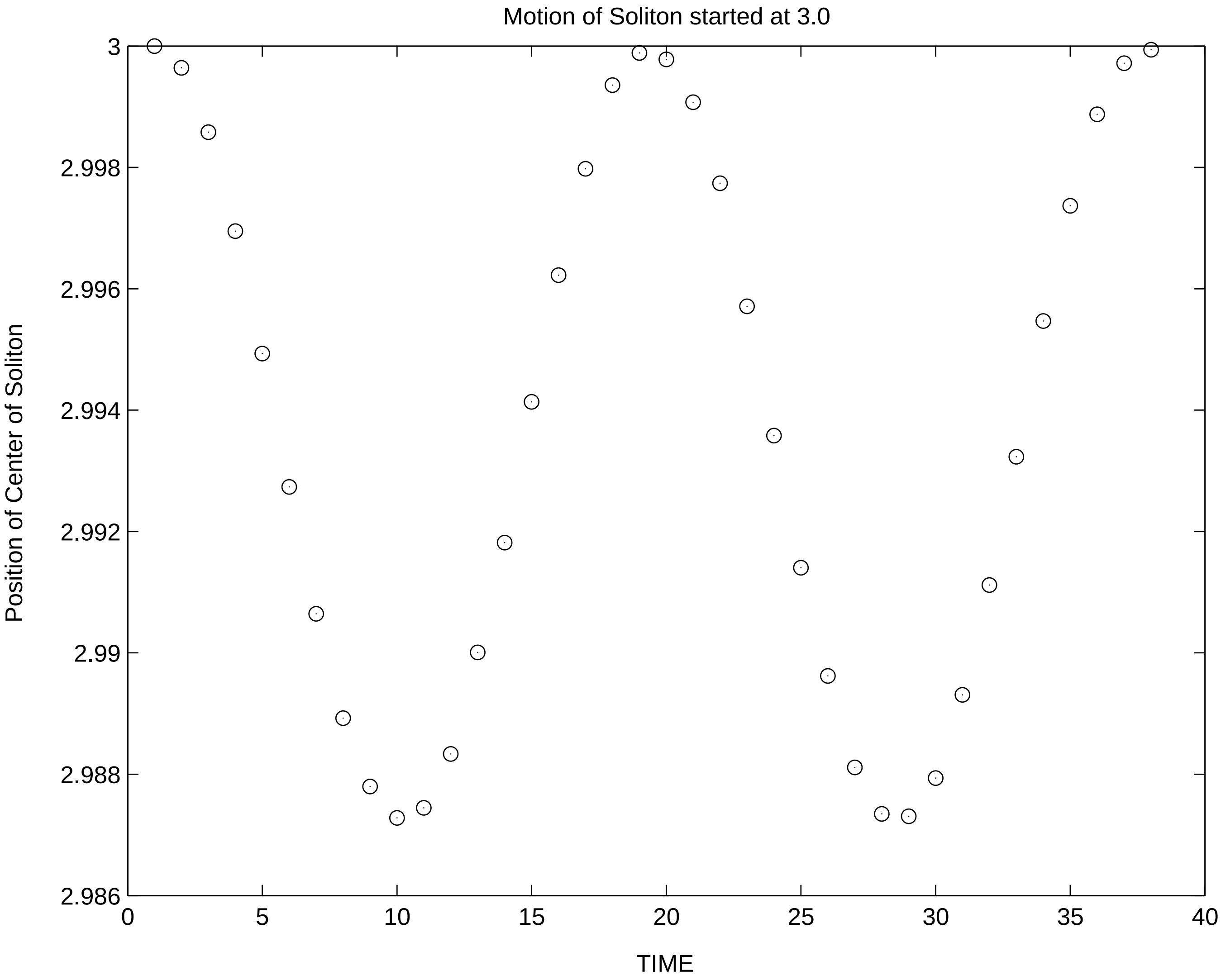

FIG. 3:

and also

$$P(x) = \alpha x(x-\beta)^2.$$

Notice that when one uses a Gaussian, that $V'''(0) = 0$ and so the soliton actually concentrates at the critical point as the theory predicts. With the use of $P(x)$, one can actually observe the soliton concentrating near but not on the critical point, again as the theory predicts. In general, a soliton with no initial velocity started at a critical point of an even potential will remain there as predicted by the theory. For an odd potential the soliton started at the critical point will begin to oscillate about the pinning point.

We include some illustrations of the dynamics of a soliton in an external potential. The graphs represent the motion of the center of a soliton with no intial velocity in the vicinity of a critical point of the potential.

In figure (2) we have a soliton whose intial position is at the critical point of a Gaussian potential given by $V(x)$. As the theory predicts, the soliton is pinned at the critical point $x = 3$. The next picture is more

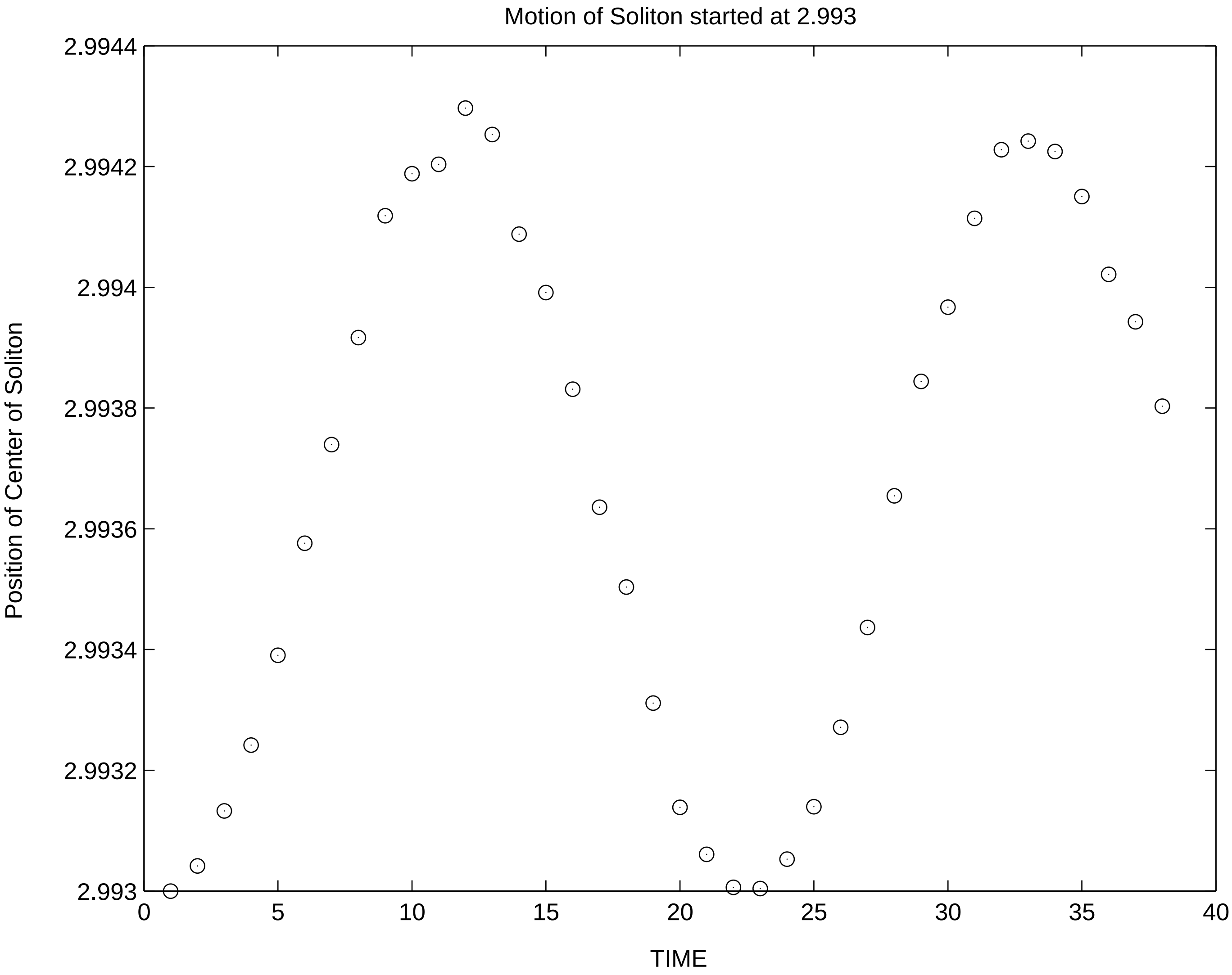

FIG. 4:

interesting. Here we have an odd potential given by $P(x)$ where we start the soliton at the critical point $x = 3$. According to the theory, the center will want to slide to the left of the critical point. What one actually observes in figure (3) is a periodic motion about some other point to the left of the critical point. In figure (4) and (5) we start the soliton nearer to this 'other point' and one can see that the oscillations become quite small. In figure (5) one actually sees that the center essentially just bounces around the 'other point', i.e. the pinning point. Finally, figure (6) represents what happens when the soliton is started at the critical point of an odd potential whose second derivative is negative. In other words the soliton is started at the top of a 'hill'. We see that this is an unstable situation and so it slides away.

We also produced simulations of a soliton with intitial velocity in the vicinity of a positive even potential and negative even potential given by $V(x)$. As expected, the soliton wobbled about the critical point of the potential when it was a mininum of the potential. Also, in the presence of a positive potential it would either

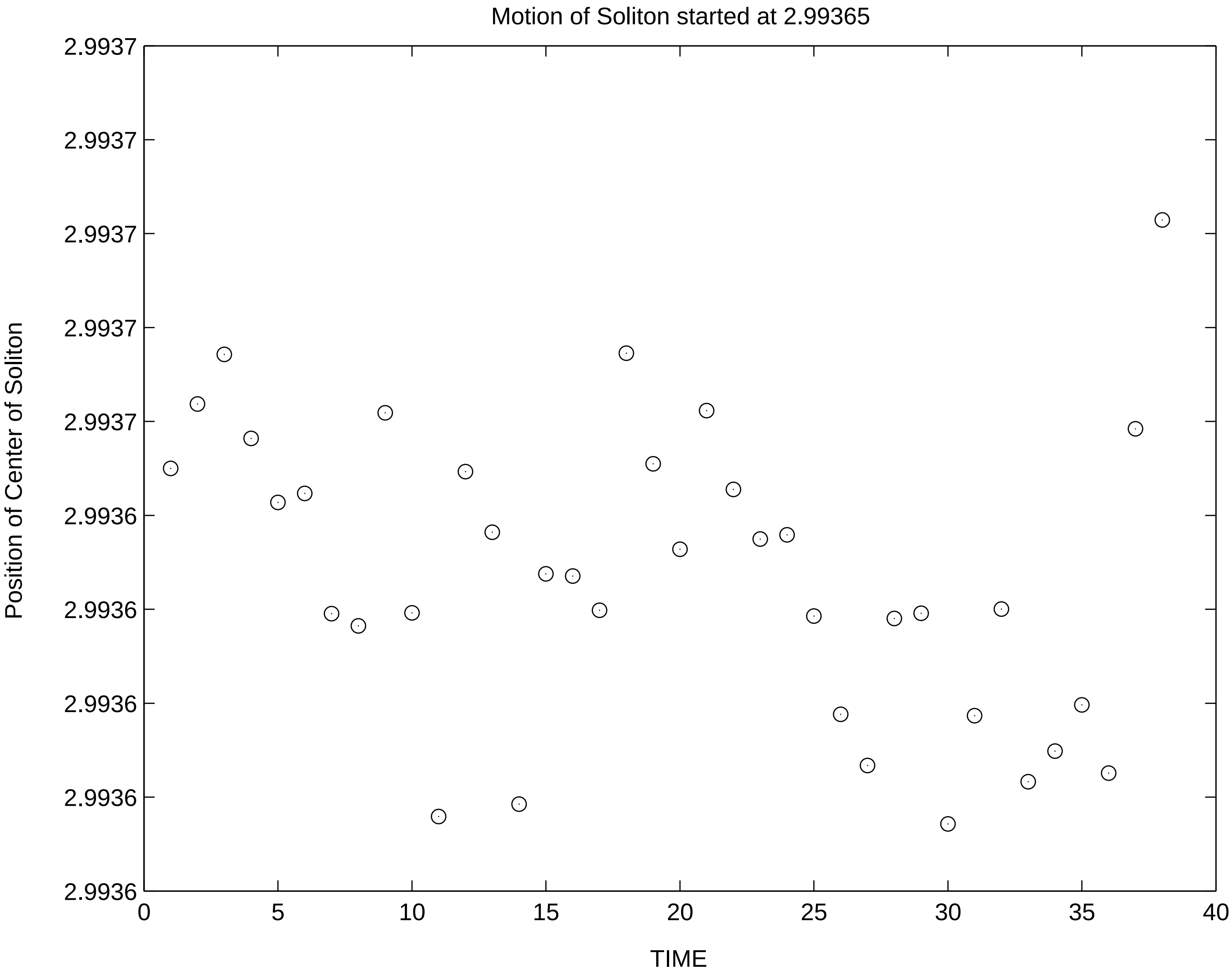

FIG. 5:

be repelled or roll over it and speed away depending on the potentials amplitude.

Many thanks to Robert Almgren and Mary Pugh, especially the latter who spent many hours educating me in the ways of mathematical computing.

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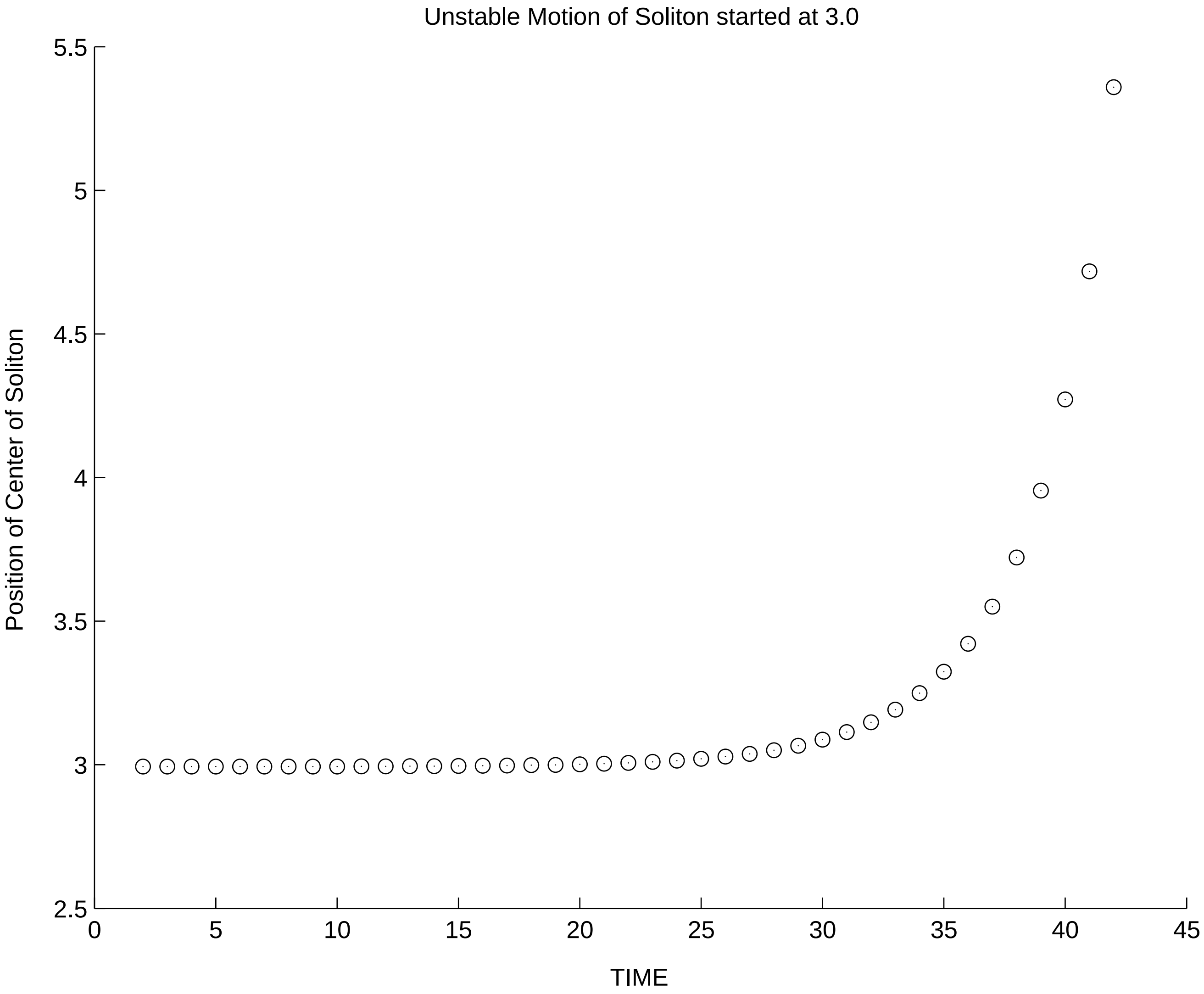

FIG. 6:

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