Research statement
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The main line of my work lies mainly in the algebraic topology of smooth group actions. More specifically, I have used tools called Borel cohomology, equivariant K-theory, and equivariant cobordism to study isotropy actions on homogeneous spaces, GKM actions, and cohomogeneity-one actions, symmetries which have been of perennial interest to differential and symplectic geometers, and to investigate questions in rational homotopy theory, including formality and toral rank. Some of this work has centered on an important algebraic condition on an action called equivariant formality. I am also interested in Lie groupoids and representations up to homotopy, and increasingly in modern homotopy theory. Besides equivariant topology, I have also worked in low-dimensional topology and dynamics [AC12, Car10], but for want of space and considerations of narrative continuity, we won’t discuss this work here.

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1. Isotropy data of stably complex manifolds

Much information about the action of a torus $T$ on a smooth manifold is encoded in the isotropy representations of $T$ on the tangent spaces of fixed points. These data uniquely determine the manifold up to equivariant cobordism and are not arbitrary, but highly interdependent, by the integral localization theorem of Atiyah–Bott–Berline–Vergne (ABBV) [BV82, AB84], which expresses this interdependency as a web of identities in the symmetric algebra on characters of $T$. The constraints imposed by these identities are so restrictive that one might also wonder if such a family of constrained representation data must necessarily arise from a torus action, so that there is a bijective correspondence between constrained abstract representation data and equivariant cobordism classes of manifolds. In the late 1990s, Viktor L. Ginzburg, Yael Karshon, and Susan Tolman asked a version of the following question.

**Realization Question.** Can any abstract isotropy data satisfying all the ABBV relations be realized as the isotropy data of some torus action on a compact, oriented, equivariantly stably complex manifold?

Elishева Adina Gamse, Karshon, and I recently proved the conjecture for an important class of well-behaved examples, the so-called GKM manifolds.

**Theorem 1.1 ([CGK18]).** Let $T$ be a torus. Given GKM abstract isotropy data $(X_p, \sigma_{p})_{p \in P}$ satisfying the ABBV relations, there exists a compact, oriented, stably complex GKM $T$-manifold $M$ with this isotropy data. In the case of a 2-dimensional torus action on a 4-manifold, there is an explicitly construction of such a manifold.

The proof uses results in equivariant cobordism, in particular unpublished work of Alastair Darby, to realize a manifold nearly realizing the given data, which we then modify via smooth equivariant surgery. In low dimensions, we can do something much more hands-on, constructing building blocks realizing representation data and equivariantly gluing them in a smooth way.

1.1. Future work: realization of abstract data in the non-GKM case

In the general case, even the statement requires some work to nail down; the constraints are a certain family of the form

$$\sum_{p \in P_{p}} \frac{\prod_{\ell \geq 0} \sigma_{\ell}(a_{i}(p))^{h_{\ell}} \prod_{k=1}^{s} \prod_{\ell \geq 0} \sigma_{\ell}(\beta_{j}(p)) : j \in J_k}{\sigma_{p} \prod \alpha_{i}(p)} = 0. \quad (1.1)$$

for every closed subgroup $H$ of $T$ and every representation $\rho$ of $H$. In this generality, the results we were able to rely on in the GKM case no longer apply and we anticipate having to dig more deeply into the cobordism theory they derive from.

2. Cohomology of homogeneous spaces

The elemental Lie group actions are the transitive actions of $G$ on an orbit $G/H$, a homogeneous space. Such a space is atomic in the equivariant realm, the analogue of a single point: more precisely, they constitute the 0-cells of a $G$–CW complex. The geometry of such a space is highly
symmetric, being identical at every point, and homogeneous spaces have long been studied by
differential geometers. Later, in Sections 4.3 and 5.2, we will discuss my results on the equivariant
cohomology and K-theory of the so-called isotropy action on a homogeneous space, but getting
a grip on this involves understanding the ordinary cohomology and K-theory of homogeneous
spaces themselves.

2.1. The cohomology of \(G/S^1\)

My thesis contains the first full proof of the following result, announced, as it turns out, by Leray
and Koszul in the late 1940s.

**Theorem 2.1** ([Car18a, Appendix A][Car15, Proposition 7.5.1]). Let \(G\) be a compact, connected Lie
group containing a circle subgroup \(S\) and \(j: G \rightarrow G/S\) the projection.

1. If \(H^1(G; \mathbb{Q}) \rightarrow H^1(S; \mathbb{Q})\) is surjective, then \(j^*\) sends \(H^*(G/S; \mathbb{Q})\) isomorphically to an exterior
subalgebra \(\Lambda \hat{P}\) of \(H^*(G; \mathbb{Q})\) such that \(H^*(G; \mathbb{Q}) \cong \Lambda \hat{P} \otimes \Lambda[z_1]\) for a generator \(z_1\) of degree 1.

2. If \(H^1(G; \mathbb{Q}) \rightarrow H^1(S; \mathbb{Q})\) is zero, then \(j^*\) sends \(H^*(G/S; \mathbb{Q})\) to an exterior subalgebra \(\Lambda \hat{P}\) of
\(H^*(G; \mathbb{Q})\) such that \(H^*(G; \mathbb{Q}) \cong \Lambda \hat{P} \otimes \Lambda[z_3]\) for a generator \(z_3\) of degree 3. There is an element
\(s \in H^2(G/S; \mathbb{Q})\) such that
\[H^*(G/S; \mathbb{Q}) \cong \Lambda \hat{P} \otimes \mathbb{Q}[s]/(s^2).\]

2.2. Monograph on the cohomology of homogeneous spaces

The background portions of my dissertation grew into a monograph [Car15] currently under
review. The book, among other things, develops the standard ("Cartan") model of \(H^*(G/K; \mathbb{Q})\)
using a streamlined version of the approach Borel developed in his thesis, without developing the
Weil algebra or rational homotopy theory. The required background is at the level of first gradu-
ate courses in algebraic topology. The necessary spectral sequences and homological algebra
are developed along the way, and the resulting exposition is substantially faster than previously
published approaches. Some aspects do not appear elsewhere in the literature.

3. Borel equivariant cohomology and equivariant K-theory

Borel cohomology has been a central tool in dealing with understanding continuous group ac-
tions since its inception in 1960. It is a well-known disappointment that the orbit space \(M/G\)
does not distinguish between isotropy types; for example, when one passes to the quotient
\(S^2/S^1 \cong [-1, 1]\) of a standard globe \(S^2\) under the action of the circle \(S^1\) by rotation, the poles
and the latitudes, which are respectively fixed points and free orbits, all become points. The only
visible difference in the quotients is that fixed points are the endpoints of the interval. One wants
to have one’s cake and eat it too by taking the quotient in a way that somehow retains the dis-

tinction between orbit types. The classic way to do this was introduced in Borel’s influential 1960
seminar on transformation groups [BBF+60]: one forms the **Borel construction**
\[M_G := EG \times M/\langle e_g, m \rangle \sim \langle e, g \rangle,\]
where $EG$ is the total space of the universal principal $G$-bundle, a contractible space with free $G$-action. Homotopically speaking, $EG \times M$ is no different than $M$, but the diagonal action on $EG \times M$ is free, so orbit types now remain distinct and we may regard $M_G$ as a homotopically-correct replacement for $M/G$. The Borel cohomology $H^*_G(M)$ of the action is the singular cohomology $H^*(M_G)$ of this new construction. For example, the homotopy quotient $(S^2)_{S^1}$ of the rotation action on the 2-sphere can be visualized as in this cartoon:

Here, forgetting the $EG$ coordinate induces a projection to the naive quotient, whose fiber over any point of the open interval $(−1, 1)$ is the (contractible) infinite-dimensional sphere $ES^1 = S^\infty$, and whose fibers over $±1$ are infinite complex projective spaces $BS^1 = \mathbb{CP}^\infty = S^\infty/S^1$. Thus $M_G$ is homotopy equivalent to the wedge sum $\mathbb{CP}^\infty \vee \mathbb{CP}^\infty$. Its cohomology $\mathbb{Z}[x, y]/(xy)$ encodes some of the structure of the action; for example, the two fixed points show up in the fact that the ring is free of rank two over the coefficient ring $H^*_S(*)$. In general, the isotropy types of orbits can be read off of the ideal structure of $H^*_G(X)$ [Hsi75, Ch. IV], so Borel cohomology makes orbit structure legible in ring theory.

Another approach to analyzing an action studies bundles over the space. Given a $G$-space $M$, one can consider the notion of an $G$-equivariant vector bundle $V \to M$ whose total space admits a $G$-action such that the projection preserves the group action. These can be directly summed and tensored just as ordinary vector bundles can, and formally inverting the direct sum yields equivariant K-theory ring $K^*_G(M)$. As in the nonequivariant case, equivariant K-theory is inherently less computable than Borel cohomology but often simpler. In the rest of this section we describe some of these computations.

3.1. ...of real Grassmannians

The real Grassmannians $G_k(\mathbb{R}^n)$ of $k$-planes in $n$-dimensional Euclidean space are important parametrizing objects, well-studied as manifolds in their own right. Accordingly, their rational singular cohomology rings have long been known [Ler49, Tak62][Car51, p. 71][Bor53, p. 192]. Chen He [He16, Thms. 5.2.2, 6.3.1, Cor. 5.2.1], applied his extension of GKM-theory to odd-dimensional and nonorientable manifolds to compute the rational Borel equivariant cohomology rings of the isotropy actions on these spaces, defined as the left multiplication action of $K$ on the right quotient homogeneous space $G/K$.

I showed [Car16], in two ways, that one can compute these rings by more direct, almost algorithmic methods, using the fact the relevant isotropy action is equivariantly formal in the sense we will discuss in Section 4; the result is actually an instance of my Theorem 4.2.
3.2. . . of cohomogeneity-one actions

The next simplest actions after homogeneous ones are the \textit{cohomogeneity-one actions}, those with one-dimensional orbit space, which are the subject of a vast geometric literature and classified in low dimensions \cite{GGZ15}. Among other things, they furnish many examples of positively-curved manifolds with large isometry group. It is natural to compute algebraic invariants of these actions, and with Oliver Goertsches, Chen He, and Liviu Mare, I computed their rational Borel cohomology. The most interesting case is the following.

**Theorem 3.1** ([CGHM18, Theorem 1.2]). Let $M$ be a smooth manifold admitting a cohomogeneity-one action of a connected Lie group $G$ with orbit space an interval, and suppose the stabilizer $H$ of a generic point in $M$ shares a maximal torus $S$ with the stabilizers $K^\pm$ of the exceptional orbits. Then there is a ring isomorphism

$$H^*_p(M; \mathbb{Q}) \cong (\text{im } \rho^*_+ \cap \text{im } \rho^*_-) \otimes H^*(S^{2\ell+1}; \mathbb{Q}),$$

where the injections $\rho^*_\pm : H^*(BK^\pm; \mathbb{Q}) \to H^*(BH; \mathbb{Q})$ are induced by subgroup inclusions and $\ell$ is a function of $\dim K^\pm/H$ and a certain dihedral subgroup of $\text{Aut } S$.

The expression was inspired by GKM-theory, which applies here if $K^\pm$ are of full rank in $G$ but not otherwise. The proof involves considerations about the action of the dihedral group on the polynomial ring $H^*_p(G/H; \mathbb{Q}) \cong H^*(BH; \mathbb{Q})$ and maps of classifying spaces, along with a novel observation about the Mayer–Vietoris sequence of the cover of $M$ pulled back from the standard two-subinterval cover of $M/G \approx [-1, 1]$.

I independently extended this work of the joint paper to equivariant K-theory. Again the result falls naturally into several cases, of which the following is the most involved.

**Theorem 3.2** ([Car18b, Theorem 4.11]). Let $M$ be a smooth manifold admitting a cohomogeneity-one action of a connected Lie group $G$ with orbit space an interval, and suppose the rank of the stabilizer $H$ of a generic point in $M$ equals that of the stabilizers $K^\pm$ of the exceptional orbits. Suppose further $K^\pm$ are products of simply-connected groups and $\text{SO}(p) \times \text{SO}(q)$ factors. Then we have a ring isomorphism

$$K^*_G M \cong (\text{RK}^-|_H \cap \text{RK}^+|_H) \otimes \Lambda[z],$$

for a generator $z$ of degree 1, where the injections $\text{RK}^\pm \to \text{RH}$ of complex representation rings are given by restriction of representations.

The proof analyzes maps of representation rings and requires some Lie theory, but as in the cohomological case uses an additional structure on the Mayer–Vietoris sequence. It turns out this extra structure obtains in great generality.

**Theorem 3.3** ([Car18b, Proposition A.11]). For any $\mathbb{Z}$-graded multiplicative $G$-equivariant cohomology theory $E^*$, the natural $E^*X$-module structure on the groups in the Mayer–Vietoris sequence of a triad $(X; U, V)$ of $G$–CW complexes with $X = U \cup V$ is preserved by the connecting map in the sequence.

This fact does not seem to appear in the literature and is needed to obtain the ring structure in Theorem 3.2. Similarly [Car18b, Lemma A.3], we are able to prove a result computing $E^*$ of a mapping torus with suitable coefficients, relying on an equivariant Atiyah–Hirzebruch
spectral sequence; this works even when there is no transfer map because the $E_2$ page is Bredon cohomology [Mat73, §4], where there is such a map.

3.3. Future work: the toral rank conjecture for nilmanifolds

A nilmanifold $N$ is a manifold which can be represented as an iterated principal torus bundle over a torus: i.e., $N$ can be written as the total space of a principal torus bundle $T \to N \to B$ where $B$ is again a nilmanifold. One can thus ask if it satisfies the following conjecture.

**Conjecture 3.4.** Let $N$ be a space of finite topological dimension admitting an action of a torus $T$ with finite stabilizers. Then $\dim \mathbb{Q} H^*(N; \mathbb{Q}) \geq \dim \mathbb{Q} H^*(T; \mathbb{Q})$.

Sullivan models are a common method of attack for this conjecture, which has been settled in certain special cases but remains open in general. The differentials in the Serre spectral sequence of $N \to B \to BT$ converging to $H^*(B; \mathbb{Q}) = H^*_T(N; \mathbb{Q})$, which are determined by certain higher cohomology operations on $H^*(N; \mathbb{Q})$ [GKM98, §13], and my collaborator Steven Amelotte and I are in the process of using these operations to establish bounds on the dimension of $H^*(N; \mathbb{Q})$ and hence verify the conjecture in this case.

3.4. Future work: representations up to homotopy as K-theory classes

Given an action of a Lie group $G$ on a space $X$, the Atiyah–Segal completion theorem [AS69], which identifies the map $K^*_G(X) \to K^*(X_G)$ with the completion of $K^*_G(X)$ with respect to the augmentation ideal $I G$ of $RG$ to be discussed in Section 5.1, can be formulated as a statement about the transformation groupoid $G \times X$. By viewing the action maps on total space $V$ and base $X$ as topological groupoids, then taking geometric realizations of topological nerves to get a vector bundle $V_G = B(G \times V) \to B(G \times X) = X_G$, we obtain a natural map

$$\text{Rep}(G \times X) \to K^*(B(G \times X))$$

that can be identified with the Atiyah–Segal map. Phrased this way, the construction generalizes to an arbitrary topological groupoid $\mathcal{G} \rightrightarrows M$, yielding an analogous correspondence²

$$\text{Rep}(\mathcal{G}) \to K^*(B\mathcal{G}),$$

$$(V \to M) \mapsto [B(\mathcal{G} s \times \pi V)].$$

One can ask if the same can be done with representations up to homotopy, generalizations necessary in the Lie groupoid world to, *inter alia*, recapture the notion of an adjoint representation [AC13]. These are conjecturally equivalent to vector bundles $E \to M$ equipped with a map $(g,e) \mapsto \lambda_g e: \mathcal{G} s \times \pi E \to E$ such that each $\lambda_g$ is linear and the differences $\lambda_g \circ \lambda_h - \lambda_{gh}$, etc., are ruled by a set of coherent homotopies. The geometric realization of the topological nerve no longer applies because the last face map $d_n: (\mathcal{G}_{s+1},e) \to (\mathcal{G},\lambda_g,e)$ only satisfies the simplicial identities up to coherent homotopy, but there is an analogous one-sided bar construction

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¹ among other things

² Cantarero [Can12] considers a related question, obtaining a completion theorem for finite $\mathcal{G}$-CW complexes and domain, instead, a $\mathcal{G}$-equivariant K-theory $K^*_\mathcal{G}(-)$ comprising classes of bundles over a $\mathcal{G}$-space $X$ which arise as summands of pullbacks, under $X \to M$ of bundles, over $M$. 
\( B(\ast, X, Y) \) for an \( A_X \)-action \( X \times Y \to Y \) [HLS16, Def. 2.8]. Applied to a representation up to homotopy, it yields a vector bundle \( B(\ast, \mathcal{G}, E) \to B(\ast, \mathcal{G}, M) \cong B\mathcal{G} \). The resulting map

\[
\text{Rep}^X(\mathcal{G}) \to K^*(B\mathcal{G})
\]

has yet to be studied.

4. Formality and equivariant formality

The homotopy quotient \( M_G \) is the total space of a fiber bundle \( M \to M_G \to BG \), where \( BG \) is the base space of the universal principal \( G \)-bundle. The fiber inclusion of this bundle induces a pullback map \( H^*_G(M) \to H^*(M) \), which in the algebraic best-case scenario is surjective. For example, in the example of \( S^1 \) rotating \( S^2 \) above, the map

\[
\mathbb{Z}[x, y] / (xy) \cong H^*_S(S^2) \to H^*(S^2) \cong \mathbb{Z}[t] / (t^2)
\]

is given by \( x \mapsto t \) and \( y \mapsto -t \). In this instance, the action of \( G \) on \( M \) is called equivariantly formal, and an element \( c \in H^*_G(M) \) carried onto \( c \in H^*(M) \) is called an equivariant extension of \( c \). While Borel already made use of this condition in his seminar, it was given its present name by Goresky, Kottwitz, and MacPherson [GKM98] in the paper that began what is now called GKM theory. This theory allows the equivariant cohomology \( H^*_G(M) \) of a a GKM manifold, one with equivariantly formal action, and finitely number of fixed points, to be computed in terms of the combinatorics of the orbits of \( o \) and \( t \)-dimensional orbits, using a lemma of Chang and Skjelbred. This is the simplifying condition figuring in Theorem 1.1. Equivariant formality guarantees all classes in \( H^*(X) \) admit equivariant extensions in \( H^*_G(X) \), to which the Atiyah–Bott–Berline–Vergne localization theorem applies, yielding the restrictions on isotropy data mentioned in Section 1.

4.1. Future work: the Weyl integral formula via ABBV

Let \( G \) be a compact, connected Lie group, \( T \) its maximal torus, \( W \) the Weyl group, and \( \Phi \subseteq \text{Hom}(T, S^1) \) the set of global roots of \( G \). If \( f : G \to \mathbb{R} \) is a continuous, conjugation-invariant function on \( G \) and \( dg \) and \( dt \) are Haar measures on \( G \) and \( T \) respectively, the Weyl integral formula states that

\[
\int_G f(g) \, dg = \frac{1}{|W|} \int_T f(t) \prod_{\alpha \in \Phi} (1 - \alpha(t)^{-1}) \, dt.
\]

On the other hand, an application of ABBV localization to the conjugation action of \( T \) on \( G \) yields a superficially similar equation

\[
\int_G f(g) \, dg = \int_T \int_{\mathcal{L}} \frac{[\text{vol}]_T}{\prod_{\alpha \in \Phi} c_1(S_\alpha)}
\]

where \( S_\alpha \) is the line bundle \( ET \times \mathbb{C}/(e, z) \sim (e, \alpha(t)z) \) over \( BT \) and the equivariant form \( [\text{vol}] \) in \( \Omega(G)^T[u_1, \ldots, u_{rkG}] \) represents a closed \( T \)-equivariant extension of the top form \( [\text{vol}] \) in \( H^*(G) \).

One might expect the former to follow from the latter. The Weyl and Kirillov character formulas are known to be essentially equivalent in the case \( G \) is compact, the former being implied by the Weyl integral formula and the latter by the the Berline–Vergne/Atiyah–Bott equivariant localization formula, so a proof may lie in analyzing this equivalence.
4.2. Future work: the Halperin conjecture for biquotients

Equivariant formality is the surjectivity of the fiber restriction $i^*$ of the bundle $M \to M_G \to BG$. One can ask the same question about other fiber bundles, and one of the main developers of rational homotopy theory, Stephen Halperin, made the following open conjecture.

**Conjecture 4.1.** Let $F$ be a simply-connected CW complex such that $\dim Q (\pi(F) \otimes Q)$ is finite and the Euler characteristic of $F$ is positive. Then for any fiber bundle $F \to E \to B$, the fiber restriction $H^*(E; Q) \to H^*(F; Q)$ is surjective.

This was verified by Shiga and Tezuka [ST87] in the case is a complete flag manifold, a homogeneous space which can be written as $G/H$ where $G$ and $H$ are connected compact Lie groups and $H$ contains a maximal torus of $G$. Their proof involved a careful analysis that, among other things, involved a standard Sullivan model for $G/H$.

I believe that the case of a biquotient $K\backslash G/H$ with $\text{rk} K + \text{rk} H = \text{rk} G$ will yield to a similar analysis using the Kapovitch model discussed in the following section 4.3.

4.3. Equivariant cohomology and K-theory of isotropy actions

Equivariant formality also simplifies computation of equivariant cohomology. The author showed the following around the time of his thesis, generalizing classical results that come to the same conclusion when $\text{rk} G = \text{rk} H$.

**Theorem 4.2 ([Car15, Theorem 10.1.1][Car18c, Theorem C]).** Let $G$ be a compact, connected Lie group, and $H$ a closed, connected subgroup such that the action of $H$ on $G/H$ is equivariantly formal. Then there is a ring isomorphism

$$H^*_H(G/H; Q) \cong H^* BH \otimes_{H^* BG} H^* BH \otimes \Lambda \hat{P},$$

where $\Lambda \hat{P} \cong \text{im}(H^*(G/H; Q) \to H^*(G; Q))$ and $H^* BH$ is an $H^* BG$-algebra through application of $H^* \circ B$ to the inclusion $H \to G$.

This result also generalizes the classical computation of $H^*(G/H; Q)$ in these cases. Our proof relies on a Sullivan model for biquotients due to Vitali Kapovitch [Kapo4, Prop. 1] which also applies to homotopy biquotients [Car16]. The model can be viewed as a compression of the Serre spectral sequence of the fibration $G \to G_{H \times H} \to BH \times BH$. Although there is no cochain-level model of equivariant K-theory, I conjectured and were eventually able to prove a related result.

**Theorem 4.3 ([Car18c, Theorem K]).** Let $G$ be a compact, connected Lie group, $H$ a closed, connected subgroup, and $k$ a subring of $Q$ such that $\pi_1 G$ is free abelian and the image of $R(G; k) \to R(H; k)$ is a polynomial ring over which $R(H; k)$ is a finite free module. Then there is a ring isomorphism

$$K^*_H(G/H; k) \cong \text{Tor}^{R_G}_*(RH, RH) \otimes k \cong RH \otimes RH \otimes \Lambda \hat{P} \otimes k,$$

where $\Lambda \hat{P} \cong \text{im}(K^*(G/H) \to K^*G)$ is an exterior algebra on $\text{rk}_Z \hat{P} = \text{rk} G - \text{rk} H$ generators and $RH$ is an $RG$-module by restriction of complex representations.

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3 We will discuss when this hypothesis is satisfied in Section 5.2.

4 and independently, much later, the present author
These conditions apply, up to taking a finite cover, in all cases where equivariant formality of an isotropy action is already known (except those I determine in Section 5.2.1). Again this result generalizes the classical computations of $K^*(G/H)$ in these cases [Min75]. The K-theoretic and cohomological results are connected by a map of spectral spectral sequences from the Künneth spectral sequence in equivariant K-theory [Hod75] to that in Borel cohomology, constructed by showing one “geometric resolution” will work for both theories and applying the equivariant Chern character, which [CF18, Thm. 5.3] identifies $H^*_G(X; \mathbb{Q})$ with the completion of $K^*_G(X; \mathbb{Q})$ with respect to $IG$ (discussed in Section 5.1). In our case of interest, $X = Y = G/H$, and the target sequence collapses, essentially because its $E_2$ term is the cohomology of the Kapovitch model, which then forces the collapse of the K-theoretic sequence.

5. Characterization of (K-theoretic) equivariant formality
Some of my work discusses equivariant formality in its own right.

5.1. Weak equivariant formality
It turns out [Fok17][CF18, Thm. 5.6] equivariant formality is equivalent rationally to surjectivity of the forgetful map $f : K^*_G(X) \to K^*(X)$ induced by discarding the $G$-structure on an equivariant bundle [MM86]. An equivariant bundle over a point is just a representation, so $K^*_G(\ast)$ is the representation ring $RG$. The trivial $G$-map $X \to \ast$ induces a map $RG \to K^*_G(X)$, and the composition with $f$ sends a representation to its dimension, annihilating the virtual representations $IG$ of dimension 0. Thus $f$ annihilates the ideal $(IG \cdot 1)$ of $K^*_G X$ and factors as

$$K^*_G X \to K^*_G X \otimes RG \overset{f}{\to} K^* X.$$  

Harada–Landweber [HL07, Prop. 4.2] observe $f$ is surjective if and only if $\tilde{f}$ is, and say that the action is weakly equivariantly formal if $\tilde{f}$ is an isomorphism. By definition, weak equivariant formality implies equivariant formality in our sense, and Fok also showed that rationally, weak equivariant formality is equivalent to equivariant formality [Fok17][CF18, Thm. 5.6]. I was able to improve this to an integral result.

**Theorem 5.1** ([Car18c, Theorem W]). If a compact, connected Lie group $G$ such that $\pi_1 G$ is free abelian acts on a compact Hausdorff space $X$ in such a way that $K^*_G X$ is finitely generated over $RG$ and the forgetful map $f : K^*_G X \to K^* X$ is surjective, then the action is weakly equivariantly formal.

The proof involves the map from the Atiyah–Hirzebruch spectral sequence of $BG$ to the Atiyah–Hirzebruch–Leray–Serre spectral sequence of $X \to X_G \to BG$, which induces a tensor decomposition that of the former which can be shown to persist to $E_\infty$.

5.2. . . . for isotropy actions
We’ve now computed the Borel cohomology and K-theory of an equivariantly formal isotropy action, so it seems only fair to say when an isotropy action is equivariantly formal.
**Question 5.2.** Let $G$ be a compact Lie group and $K$ a closed subgroup. When is the isotropy action of $K$ on $G/K$ equivariantly formal?

At the beginning of 2014, only three classes of examples were known: generalized flag manifolds, those for which $H^*(G; \mathbb{Q}) \longrightarrow H^*(K; \mathbb{Q})$ is surjective, and *generalized symmetric spaces* [GN16]. In collaboration with Fok, the author was able to extend this to a complete characterization.

**Theorem 5.3** ([CF18, Thm. 1.4, Prop. 3.13]). Let $G$ be a compact, connected Lie group and $K$ a closed, connected subgroup. The following are equivalent:

1. $K$ acts equivariantly formally on $G/K$;
2. $G/K$ is a rationally formal space and $H^*(BG; \mathbb{Q}) \longrightarrow H^*(BK; \mathbb{Q})^{N_c(K)}$ is a surjection;
3. $\pi_0 N_G(S)$ acts on the tangent space $s$ to a maximal torus $S$ of $K$ as a reflection group and $H^*(BS; \mathbb{Q})^{N_c(S)}$ is a surjection.

The main work was in proving formality in (1) $\implies$ (2), which again required analysis of the aforementioned Kapovitch model. The rest involved a refinement of the proof of a theorem of Shiga–Takahashi [Shig6, Thm. A, Prop. 4.1] [ST95, Thm. 2.2] relaxing its assumptions. The reflection group statement follows from the Chevalley–Shepherd–Todd theorem [Kan94, p. 82].

5.2.1. ... of rank one

This follows on a string of reductions established in my dissertation [Car18a, Car15].

**Theorem 5.4.** Given a compact, connected Hausdorff (respectively, connected Lie group) $G$ and a closed, connected Lie subgroup $H$, the question of whether the isotropy action of $H$ on $G/H$ is equivariantly formal reduces to the same question about a quotient Lie group (resp., maximal compact subgroup) $K_G$ and the maximal torus $S$ of $K_G$, the image of (resp., intersection with) $H$ in $K_G$. If $\pi_0 N_G(S) \cong \pi_0 N_G((S \cap G')_0)$, there is a further reduction to $\tilde{K}_G'$, a simply-connected group (the universal cover of the commutator subgroup $K_G'$) and $(S \cap G')_0$ a torus in $\tilde{K}_G'$.

For circles, I found a complete classification.

**Theorem 5.5.** For $K = S^1$, there is an explicit algorithm determining whether the isotropy action of $K$ on $G/K$ is equivariantly formal. If $K \in \{SU(2), SO(3)\}$, then $K$ acts equivariantly formally on $G/K$ for any Lie supergroup $G$ of $K$.

The algorithm examines projections $K_j \cong S^1$ in simple factors $G_j$, using Lie theory to determine whether in each $G_j$ there is some $g$ such that $g z g^{-1} = z^{-1}$ for $z \in K_j$; The action is equivariantly formal just if this is so for all $j$. The most complicated possibility is $G_j = E_6$, where $S^1$ acts on $E_6/S^1$ equivariantly formally if $S^1$ lies in a Spin(8) subgroup. In any given maximal torus, it turns out, $S^1$ must then lie in one of 45 identifiable 4-dimensional subtori spanned by mutually orthogonal quadruples of roots.
5.3. Future work: corank one

The fact that it is possible to determine explicitly when a circle $S^1$ in a Lie group $G$ is such that its isotropy action on $G/S^1$ is equivariantly formal leads one to ask if a similar classification is possible for tori $S$ of codimension 1 in a maximal torus $T$. This is the subject of current joint work with Chen He. One of the main tools seems to be the circle bundle $T/S \rightarrow G/S \rightarrow G/T$, whose Euler class determines the cohomology of $G/S$, and another key piece seems to be representation of the reflection group $\pi_0 N_G(S)$ on the Lie algebra $\mathfrak{s}$, as expected from the last condition in Theorem 5.3.

References


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