

## Fibering Manifolds-with-Boundary

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$(M, \partial M)$  manifold-with-boundary

$K$  = sectional curvature of  $M$

$II$  = second fundamental form of  $\partial M \hookrightarrow M$

$(II(X, Y) = g(\nabla_X \nu, Y))$  where  $\nu$  is normal, and  $X, Y \in T^1 \partial M$ .

$0 \leq II$  means  $\partial M$  is locally convex.

$\lambda^- \leq II \leq \lambda^+$  means  $\partial M$  can be neither too concave nor too convex.

**Example.** *The standard closed, flat disc  $D^2(r)$  of radius  $r$  has  $II \equiv \frac{1}{r}$ .*

$X, Y$  metric spaces.

**Definition.** the Gromov-Hausdorff distance is  
 $d_{GH}(X, Y) = \inf\{\epsilon > 0 : \text{there exist metric space } Z$   
and isometric embeddings  $i_X : X \hookrightarrow Z$  and  $i_Y : Y \hookrightarrow Z$  with  $i_Y(Y) \subseteq B(i_X(X), \epsilon; Z)$  and  $i_X(X) \subseteq B(i_Y(Y), \epsilon; Z)\}$

**Definition.** an  $\epsilon$ -Hausdorff approximation  $f : X \rightarrow Y$  is a (not nec. cont.) map s.t.  
(i)  $f(X)$  is an  $\epsilon$ -net in  $Y$ , i.e.,  
 $B(f(X), \epsilon; Y) = Y$  and  
(ii)  $f$  is an  $\epsilon$ -almost isometry, i.e.,  
 $|d_Y(f(x_1), f(x_2)) - d_X(x_1, x_2)| \leq \epsilon \quad \forall x_1, x_2 \in X$

$d_{GH}(X, Y) \leq 3\epsilon$  if there exists an  $\epsilon$ -Hausdorff approximation  $f : X \rightarrow Y$ .

**Definition.** a geodesic terminal of a space  $X$  is a point  $x \in X$  s.t. some (nontrivial) geodesic  $\gamma : [0, L] \rightarrow X$  with  $\gamma(L) = x$  is not contained in a geodesic  $\tilde{\gamma} : [0, L + \epsilon] \rightarrow X$  with  $\epsilon > 0$ .

**Definition.** a space  $X$  is geodesically extendible if it contains no geodesic terminals.

**Example.** in a manifold-with-boundary, the boundary comprises the set of geodesic terminals.

**Example.** for a cone over a circle of circumference less than  $2\pi$ , the vertex is a geodesic terminal.

## Two fundamental questions

how geodesic terminals may appear (in a sequence of closed manifolds): only if  $inj \rightarrow 0$

how they may disappear (from a sequence of singular spaces): necessary conditions unknown

**Example.** Let  $M_i :=$  portion of  $\mathbb{R}^3$  bounded by the graphs of  $f_0(x, y) = e^{-(x^2+y^2)}$  and  $f_i(x, y) = e^{-(x^2+y^2)} + \frac{1}{i}$ . Then  $K \equiv 0$ ,  $|II| \leq \lambda$  for some  $\lambda < \infty$ , and the limit  $X$  has  $curv X \geq -\lambda^2$ .

**Example.**  $M_i := S^1(\frac{1}{i}) \times D^1(\frac{1}{i}) \xrightarrow{GH} pt$ , under  $K \equiv 0$ ,  $II \equiv 0$ ,  $inj(M_i) = \frac{\pi}{i} \rightarrow 0$ .

**Theorem. (Cheeger, Gromoll 1972)** *Let  $(M, \partial M)$  be a complete Riemannian manifold. If  $0 \leq K_M$ ,  $0 \leq II_{\partial M}$  then  $M$  is diffeomorphic to the normal bundle over a closed totally geodesic submanifold.*

proof employs a finite number of maximal inward equidistant retractions, each maintaining locally convex boundary and dropping the dimension, until there is no more boundary.

**Theorem** (Yamaguchi, 1994). *If  $(M_i, \partial M_i) \xrightarrow{GH} N$  where  $N$  is a closed manifold,  $K^- \leq K_{M_i}$ ,  $0 \leq II_{\partial M_i}$ , then  $F \rightarrow M_i \rightarrow N$  is a locally trivial fiber bundle, where  $F$  is a manifold-with-boundary.*

**Fibering-Capping Theorem**(Yamaguchi, 2000)  
*If  $M_i$  is a sequence of closed manifolds converging to a compact manifold-with-boundary  $(N, \partial N)$  under  $K_{M_i} \geq K^-$ , then for large  $i$ ,  $M_i = M_{int} \cup M_{cap}$ , where  $M_{cap}$  is a manifold-with-boundary, and  $M_{int} \rightarrow int N$  and  $M_{cap} \rightarrow \partial N$  are locally trivial fiber bundles.*

**Theorem** (Kodani, 1990).

*In  $\mathcal{M}(n, K^\pm, \lambda^\pm, i_\partial, i_{int})$ , the Gromov-Hausdorff and Lipschitz topologies coincide, i.e., for all  $\delta > 0$  there exists  $\epsilon > 0$  such that for any manifolds-with-boundary  $M, N \in \mathcal{M}$ ,  $d_{GH}(M, N) < \epsilon$  implies  $d_L(M, N) < \delta$ .*

Remark: the hypotheses imply that manifolds-with-boundary in  $\mathcal{M}$  have a uniform lower volume bound.

**Fibering Theorem. (W.)** Suppose

$(M_i, \partial M_i) \xrightarrow{GH} X$ , with either:

- (i)  $X$  a compact Poincaré duality space, or
- (ii)  $X$  geodesically extendible.

*If  $|K_{M_i}| \leq K$ ,  $|II_{\partial M_i}| \leq \lambda$  and  $inj(M_i) \geq i_0 > 0$ , then  $D^1 \rightarrow M_i \rightarrow X$  is a locally trivial fiber bundle.*

**Definition.** For a geodesic metric space  $X$ , and  $x \in X$ , let  $inj(x) = \sup\{r > 0 : \text{any minimizing geodesic issuing from } x \text{ is unique up to distance } r\}$ . Define  $inj(X) = \inf_{x \in X} \{inj(x)\}$ , the intrinsic injectivity radius of  $X$ .

**Definition.** For a manifold-with-boundary  $(M, \partial M)$ , the inradius is  $inrad(M) = \sup_{p \in M} \{inrad(p)\}$ , where  $inrad(p) = \sup\{r > 0 : B(p, r) \subseteq intM\}$ .

**Example.**  $S^1(r) \times D^1(\epsilon)$  has  $inrad = \epsilon$ ,  $inj = \pi r$ .

$D^2(r)$  has  $inrad = r$ ,  $inj = \infty$ .

$\mathbb{R}^2 \setminus B^2(r)$  has  $inrad = \infty$ ,  $inj = \pi r$ .

$(M, \partial M)$ , as shown, has  $inrad = r$ ,  $inj = \infty$ .

main idea of proof of Fibering Theorem: show inradius tends to zero, then invoke

**Thin-Manifolds-with-Boundary Theorem (Alexander, Bishop 1998)** *There is a constant  $c (\geq 0.075)$  s.t. if  $M$  is a complete manifold satisfying*  

$$\text{inrad}(M) \cdot \max\{\sup |K_M|, \sup |II|^2\} < c^2,$$
*then either  $M$  is diffeomorphic to the product of a manifold without boundary and an interval or  $M$  can be doubly covered by such a product.*

Proof of Fibering Theorem,  
locally convex boundary case:

**Proposition (W.)** (No long segments uniformly transverse to the boundary)

*Let  $(M_i, \partial M_i)$  have locally convex boundary and  $K^- \leq K_{M_i}$ . Assume  $M_i \xrightarrow{GH} X$ , where  $X$  is geodesically extendible. Then for all  $R > 0, \theta > 0$ , there exists an  $i_0$  such that for all  $i \geq i_0$ ,  $M_i$  admits no minimal segment  $[q_i p_i]_{M_i}$  of length  $\geq R$  with  $p_i \in \partial M_i$  and  $\angle^{M_i}([p_i q_i]'_{M_i}, \partial M_i) \geq \theta > 0$ .*

*Proof:*

- ①  $p_i \in \partial M_i$
- ②  $\angle([q_i p_i], \partial M_i) \geq \theta > 0$  with  $|q_i p_i| = R > 0$
- ③  $[q_i p_i] \rightarrow [qp]$
- ④ extend  $[qp]$  to  $[qpr]$  with  $|pr| = \delta > 0$
- ⑤  $\exists r_i \rightarrow r$

Then

$$(1) \angle^{M_i} q_i p_i r_i \leq \pi - \theta \quad \forall i$$

$$(2) \overline{Z}_k^X qpr = \angle^X qpr = \pi$$

$$(3) \angle^{M_i} q_i p_i r_i \geq \overline{Z}_k^{M_i} q_i p_i r_i \text{ since } \text{curv} M_i \geq k$$

$$\text{Now } |q_i p_i| \longrightarrow |qp| = R,$$

$$|p_i r_i| \longrightarrow |pr| = \delta,$$

$$\text{and } |q_i r_i| \longrightarrow |qr| = R + \delta$$

imply

$$(4) \overline{Z}_k^{M_i} q_i p_i r_i \longrightarrow \overline{Z}_k^X qpr$$

But (2),(3),(4) together contradict (1). So  $R = 0$ ,  $i_{\partial}(p_i) \longrightarrow 0$  for any boundary point  $p_i \in \partial M_i$ , and  $\text{inrad}(M_i) \longrightarrow 0$ .  $\square$

**Corollary.** Suppose  $(M_i, \partial M_i)$  satisfy  $0 \leq K_{M_i}$ , and  $0 \leq II_{\partial M_i}$ . Assume  $M_i \xrightarrow{GH} N$ , where  $N$  is a closed manifold.

Then  $F \longrightarrow M_i \longrightarrow N$  is a locally trivial fiber bundle, where the fiber  $F$  is an almost non-negatively curved (in the generalized sense)  $D^k$ -bundle over a closed manifold ( $1 \leq k \leq n$ )

Proof of general case (of the Fiberings Theorem):

uses:

- Alexandrov extension
- injectivity radius estimates, to transfer “injectivity” from  $M$  across  $\partial M$  to the collar side
- arc/chord comparison

## Alexandrov spaces

**Definition/Theorem.** *a geodesic metric space  $X$  is an Alexandrov space of curvature bounded below by  $k \in \mathbb{R}$  ( $\text{curv}X \geq k$ ) if for any geodesic triangle  $\Delta xyz$  in  $X$ , there exists a comparison triangle  $\bar{\Delta}xyz$  such that  $t \geq \bar{t}$ .*

**Example.** *Any closed manifold  $M$  with sectional curvature  $K_M \geq k$*

**Fact.** *If  $\text{curv}X_i \geq k$  and  $X_i \xrightarrow{GH} X$ , then  $\text{curv}X \geq k$*

## Alexandrov Extension

Fix  $n \geq 2$  and  $K^-, \lambda^\pm \in \mathbb{R}$ . Let  
 $\mathcal{M}(n, K^-, \lambda^\pm) \equiv \{M \text{ Riemannian } n\text{-mfld} : K^- \leq K_M, \lambda^- \leq II_{\partial M} \leq \lambda^+\}$ .

**Extension Proposition (W.)** For any  $M \in \mathcal{M}(n, K^-, \lambda^\pm)$  there exists a locally isometric, uniform extension  $\tilde{M}$  of  $M$  which is an Alexandrov space of curvature bounded below by  $c(K^-, \lambda^\pm)$ .

Uniform means  $d(\partial M, \partial \tilde{M})$  is no smaller than a constant which may be chosen arbitrarily.

$\tilde{M} = M \cup C_M$ , where  $C_M$  is produced in the following

**Lemma.** For any  $M \in \mathcal{M}(n, K^-, \lambda^\pm)$ , there exists an intrinsic metric on  $C_M := \partial M \times [0, t_0]$ , for some  $t_0$ , such that

- $II_{\partial M \times \{0\}} \geq |\min\{0, \lambda^-\}| \geq 0$ ,
- $II_{\partial M \times \{t_0\}} \equiv 0$ , and
- the sectional curvature of  $\partial M \times [0, t_0]$  is bounded below by  $k(K^-, \lambda^\pm)$ .

Properties:

- $\widetilde{M}$  is a  $C^{1,\alpha}$  differentiable manifold with  $C^0$  Riemannian structure, and a.e.  $C^\infty$ .
- $\widetilde{M}$  is realized as the  $W_{loc}^{2,p}$ -limit of  $C^\infty$  smooth manifolds, having lower curvature bounds approaching that of  $\widetilde{M}$ .

These in turn are constructed using Sobolev averaging of the metrics on both sides of the gluing locus.

**Theorem** (Kossovski 2001).

*If  $(M_1, \partial M_1)$  and  $(M_2, \partial M_2)$  have interior sectional curvatures bounded below by  $k$ ,  $\partial M_1 = \partial M_2$ , and the sum of the second-fundamental forms w.r.t. respective normals is positive semi-definite, then  $\text{curv}(M_1 \cup M_2) \geq k$ .*

## Arc/chord Comparison

$U$  a  $CAT(K)$ -domain

$\gamma : [0, 1] \rightarrow U$  a curve with arc/chord curvature  $\kappa \leq k$ .

$s$  = arclength

$r$  = chordlength

**Theorem** (Alexander, Bishop 1996). *Assume  $s + r < 2\pi/\sqrt{K}$ . Let  $\mu$  be the complete  $k$ -curve in the model space  $M_K^2$ .*

*If  $r < \text{diam}(\mu)$  and  $\text{diam}(\gamma) < \text{diam}(\mu)$  then  $\gamma$  is no longer than the minor arc of  $\mu$  with chordlength  $r$ .*

Let  $s = s(r, k, K)$  be the arclength of a minor  $k$ -arc of chordlength  $r$  in  $M_K^2$ . Then

$$s = r + \frac{1}{24}k^2r^3 + \frac{9k^4 + 8k^2K}{1920}r^5 + \dots$$

### Base-Angles

**Theorem** (Alexander, Bishop 1996). *If  $\gamma$  is a curve in  $U$  with  $\kappa \leq k$  and length no more than half a complete  $k$ -curve in  $M_{\bar{K}}^2$ , then the base-angles of  $\gamma$  are no greater than they are for a  $k$ -arc  $\bar{\gamma}$  in  $M_{\bar{K}}^2$  of the same length.*

In symbols,

$$\angle(\gamma, [\gamma(0)\gamma(1)]) \leq \angle(\bar{\gamma}, [\bar{\gamma}(0)\bar{\gamma}(1)]) = kr/2 + O(r^3),$$

where  $r = |\gamma(0)\gamma(1)|$  denotes chordlength.

**Main Angle Lemma. (W.)** *For  $\tilde{M}$  an Alexandrov extension of  $M$ ,*

$$|\angle^M xyz - \angle^{\tilde{M}} xyz| \leq \tau_{K^\pm, \lambda^\pm, i_0}(R)$$

*if  $x, y, z \in M$ ,  $|xy|, |yz| \leq R$ ,  $y \in \partial M$ ,  
 $[xy]_M \perp \partial M$ .*

Here  $\tau(R)$  denotes a function which tends to 0 as  $R \rightarrow 0$

idea of the proof:  $\tilde{M} = M \cup C_M$  will not have  $\text{curv} \tilde{M} \leq k$  in general, but can apply arc/chord comparison to  $M$  and  $C_M$  separately.

**Definition** (Lytchak). A subspace  $Z$  of a length space  $X$  is called  $(C, \rho)$ -convex if for all  $w \in Z$  and any  $x, y \in B(w, \rho(w); X) \cap Z$ , the metrics satisfy  $d_Z(x, y) \leq d_X(x, y) + Cd_X^3(x, y)$ .

here  $\rho > 0$  is a function on  $X$

For instance, a 0-convex subset is simply a locally convex subset.

Under the hypotheses of the Fiberings Theorem, both  $M$  and  $\partial M$  are  $C$ -convexly embedded in  $\widetilde{M}$ :

**Lemma.**  $|xy|_{\widetilde{M}} \leq |xy|_M \leq |xy|_{\widetilde{M}} + C|xy|_{\widetilde{M}}^3$  for all  $x, y \in M$  sufficiently close, where  $C = C(K^\pm, \lambda^\pm, i_0) \geq 0$   
and  $|xy|_{\widetilde{M}} \leq |xy|_{\partial M} \leq |xy|_{\widetilde{M}} + C|xy|_{\widetilde{M}}^3$  for all  $x, y \in \partial M$  sufficiently close, where  $C = C(K^\pm, \lambda^\pm, i_0) \geq 0$

The proof the Main Angle Lemma also uses:

**Proposition. (W.)** *Suppose  $\text{curv}X \geq k$ . Let  $Z \subset X$  be  $C$ -convex, for some constant  $C \geq 0$ . Then  $\angle^Z(\gamma_1, \gamma_2) = \angle^X(\gamma_1, \gamma_2)$  for all  $Z$ -geodesics  $\gamma_1$  and  $\gamma_2$ .*

**Corollary.** *Let  $X$  and  $Z$  be as in the proposition, with  $(n, \delta)$ -str.rad( $X$ )  $\geq i_0 > 0$  (for some  $\delta < \pi/4$ , say; in particular,  $X$  is weakly geodesically extendible). Then  $Z$  has no  $C^1$ -smoothly closed geodesic of length less than  $i_1 = i_1(C, k, i_0) > 0$ .*

**Proposition.** *Let  $Z \subseteq X$ ,  $\text{curv}X \geq k$ ,  $\text{curv}Z \geq k$ ,  $X$  geodesically extendible,  $d_Z \leq d_X + Cd_X^3$  locally. Then*

$$\angle^X([xy]_X, [xy]_Z) \leq \tau_C(|xy|_X)$$

*for all  $x, y \in Z$ .*