Lecture 9 (?)

$$S(t) = \sum_{\lambda \in Sp_\Delta} e^{\pm i\sqrt{\lambda}\, t} = \sum_{\gamma \text{ periodic geodesic}} E_\gamma(t) + E_0(t)$$

nnp

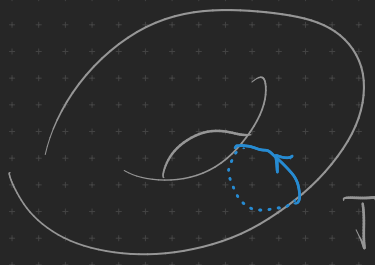
Distributions with "bias" sm $t=0$

$E_\gamma(t)$ is smooth unless $t = \operatorname{len}\gamma = T_\gamma$

Duistermaat + Guillemin:

Prim. period of γ

$$E_\gamma(t) = \frac{1}{t - T_\gamma} \cdot \frac{T_{0\gamma}}{2\pi} \, i^{\sigma_\gamma} \frac{1}{\sqrt{|\det I - P_\gamma|}}$$



$+$ ho. sing. terms.

$T_{0\gamma}$, $T_\gamma \in \mathbb{Z} \cdot T_{0\gamma}$

$$S(t) = \operatorname{Tr} U(t) \qquad [U(t)v](x) = \int K(t,x,y)\, v(y)\, dy$$

where $K(t;x,y) = \sum_{n \in \mathbb{N}} e^{i\sqrt{\lambda_n}\, t} \cdot e_n(x)\, \overline{e_n(y)}$

$$= \int_M K(t,x,x)\, dx$$

Intuition: if we were in a flat metric

$$k(t,x,y) = \int e^{i\,2\pi\|\xi\|t - \langle \xi, x-y\rangle}\, d\xi$$

$\hat{\xi}\, t - (x-y) = 0$

in general (NOT AT ALL OBVIOUS)

$$k(t,x,y) = \sum_\ell \int e^{i\,\phi_\ell(t,x,y,\xi)}\, a_\ell(x,\xi)\, d\xi$$

"phase" (smooth, real), 1-homogeneous in ξ.

smooth cpt supported function.

Oscillatory integrals

$$I(\tau) = \int e^{i\tau\phi(x)}\, a(x)\, dx$$

a: cpt support

1. if $\nabla\phi \neq 0$ in some nbhd of $\operatorname{supp} a$, then I has rapid decay $\|t^N I\|_\infty < \infty \ \forall N$.

2. if $\nabla\phi(p) = 0$ for $p \in \operatorname{supp} a$

Thm Let ϕ be smooth, $a \in C_0^\infty$. Let $p_0, \dots, p_{n-1}$ be crit pts of ϕ (i.e. $\nabla\phi(p_\ell) = 0 \quad \ell = 0, \dots, n-1$) and:

$$\det\left(\operatorname{Hess}\phi\big|_{p_\ell}\right) \neq 0$$ non-deg assumption

Γ_ℓ = signature of $\operatorname{Hess}\phi|_{p_\ell}$ then, as $\tau \to \infty$

$$I(\tau) \sim \frac{(2\pi)^{n/2}}{\tau^{n/2}} \sum_{\ell=0}^{n-1} \frac{e^{i\pi\sigma_\ell/4}}{\sqrt{|\det \mathrm{Hess}\,\phi|_{p_\ell}|}} e^{i\tau\phi(p_\ell)} \left[a(p_\ell) + \sum_{j=1}^{\infty} \left[\alpha_j^\ell(a)\right](p_\ell)\, \tau^{-j} \right]$$

Differential operators of order $2j$

Pf: Assume $n=1$, $p_0 = 0$. $\mathrm{Hess}\,\phi|_{p_0} = A$

$$\int e^{i\tau\left(\frac{x^T A x}{2}\right)} a(x)\,dx \qquad A \in M^{n\times n}$$

Assume A is diagonal (by density)

$$\int e^{\frac{i\tau}{2}\left(x_1^2\lambda_1 + x_2^2\lambda_2 + \cdots x_n^2\lambda_n\right)} a(x)\,dx$$

$$a(x) = \sum_u a_{i_1}^{u}(x_1)\cdot a_{i_2}^{u}(x_2)\cdots a_{i_n}^{u}(x_n)$$

by density argument I can choose

$n=1$

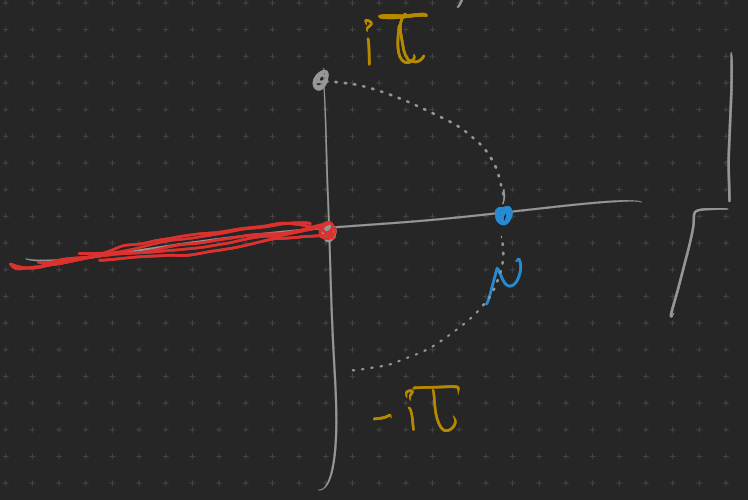
$$\int e^{i\tau\frac{x^2\lambda}{2}} a(x)\,dx$$

$\lambda \neq 0$ $\begin{cases} \lambda < 0 \\ \lambda > 0 \end{cases}$.

Assume $\lambda < 0$, for any $\nu > 0$ let $\left(\int \hat{f}(x) g(x) = \int f(w) \hat{g}(w) \right)$

$$J(\nu) = \int e^{\nu \frac{x^2}{2} \lambda} a(x)\, dx = \frac{1}{\sqrt{-\nu\lambda}} \int e^{\frac{\omega^2}{2\lambda\nu}} \hat{a}(\omega) d\omega$$

Observe that $J(\nu)$ is analytic in ν



$$\frac{e^{-i\pi/4}}{\sqrt{\tau}\sqrt{\lambda}} \int e^{\frac{\omega^2}{2\lambda\nu}} a(\omega) d\omega$$

$$e^{\frac{\omega^2}{2\lambda\nu}} = 1 + \frac{\omega^2}{2\lambda\nu} + \frac{1}{2} \frac{\omega^4}{4\lambda^2\nu^2} + \frac{1}{\cdots} + \cdots$$

$$\int a(\omega) + \frac{d}{d\omega} \int \frac{i\omega^2}{2\lambda\tau} a(\omega) d\omega$$

$$\frac{e^{(-\operatorname{sign}\lambda) i\pi/4}}{\sqrt{\tau}\sqrt{|\lambda|}} \left[a(0) + \sum \frac{c_j}{\tau^j} a^{2j}(0) \right]$$

$$\frac{e^{i\pi/4\, \sigma_A}}{\tau^{n/2} \sqrt{|\det A|}} \exp\left(\text{jet } a \,|_0 \right)$$

□