Lecture 4

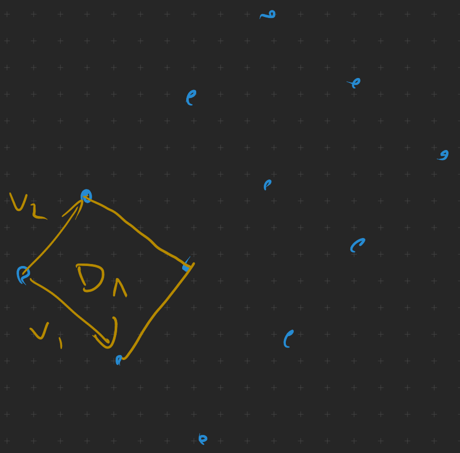
Trace formulas for $\mathbb{T}_\Lambda$

- $S(\mathbb{R}^n;\mathbb{C})$ Schwartz functions
- Fourier transform $\forall f \in S(\mathbb{R}^n)$

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i2\pi\langle \xi, x\rangle} f(x)\,dx$$

$$\check{f}(x) = \int_{\mathbb{R}} e^{i2\pi\langle \xi, x\rangle} f(\xi)\,d\xi$$

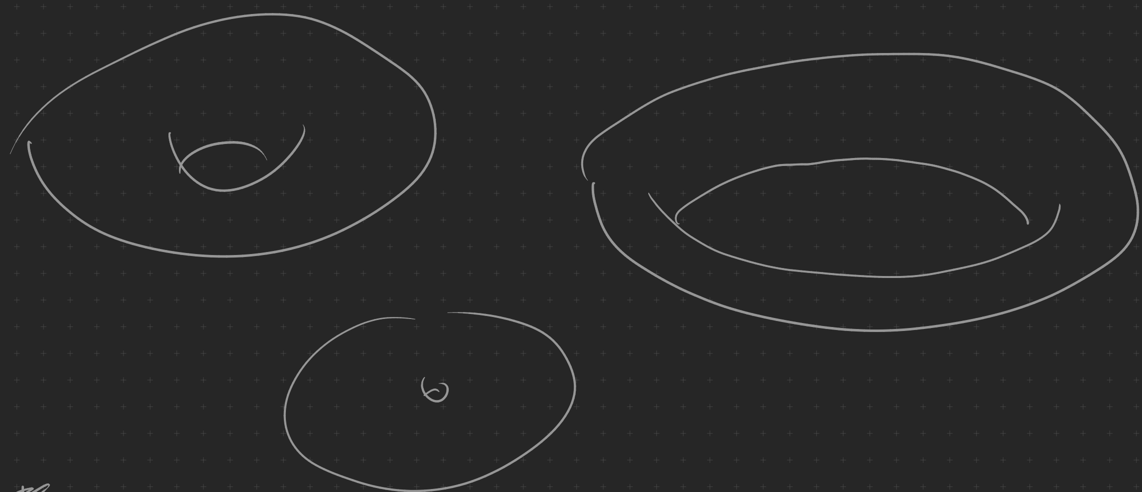
- Λ full-rank lattice span $\Lambda = \mathbb{R}^n$
  $\exists$ iso $\phi : \mathbb{Z}^n \to \Lambda$



$D_\Lambda/\sim$ ← parallel sides are glued $= \mathbb{T}_\Lambda$

$\parallel$

$\mathbb{R}^n/\sim_\Lambda$ $x \sim_\Lambda y \iff x - y \in \Lambda$



- $\Lambda^* = \{ \xi \in \mathbb{R}^n \mid \langle \xi, x\rangle \in \mathbb{Z} \ \forall x \in \Lambda \}$
- $\mathrm{covol}\,\Lambda := \mathrm{Vol}\, D_\Lambda = \mathrm{Vol}\,\mathbb{T}_\Lambda = (\mathrm{covol}\,\Lambda^*)^{-1}$

Poisson's summation formula $\forall f \in S(\mathbb{R}^n)$

$\forall$ full-rank $\Lambda \subset \mathbb{R}^n$

$$\sum_{\xi \in \Lambda^*} f(\xi) = \operatorname{covol} \Lambda \sum_{s \in \Lambda} \check{f}(s) \quad (*)$$

Heat trace :

Apply $(*)$ to $f(x) = e^{-\alpha \pi \|x\|^2}$

Fact : $\check{f}(\xi) = \alpha^{-n/2} e^{-\frac{\pi}{\alpha}\|\xi\|^2}$

Choose $\alpha = 4\pi \cdot t$ where $t \in \mathbb{R}_{>0}$

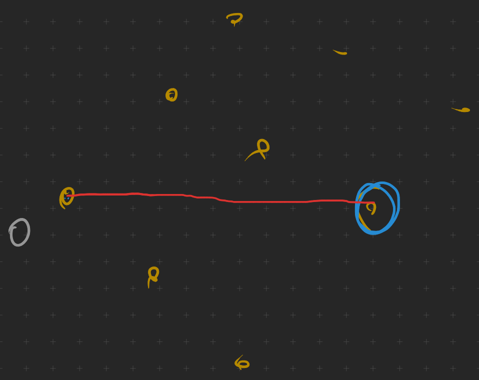
$$\sum_{\xi \in \Lambda^*} e^{-4\pi^2 \|\xi\|^2 t} = \frac{\operatorname{covol} \Lambda}{(4\pi t)^{n/2}} \sum_{s \in \Lambda} e^{-\frac{\|s\|^2}{4t}}$$

Recall : $Sp_\Delta(\mathbb{T}_\Lambda) = \{ 4\pi^2 \|\xi\|^2 \mid \xi \in \Lambda^* \}$

So $LHS = \sum_{\lambda \in Sp_\Delta(\mathbb{T}_\Lambda)} e^{-\lambda t}$, it is called the Heat trace as $LHS = \operatorname{Tr}\left[e^{-\Delta t} \right]$

Recall! for $\mathbb{T}_\Lambda$, periodic geodesics correspond to $s \in \Lambda$

$$\widetilde{LS}(\mathbb{T}_\Lambda) = \left(\|s\| \right)_{s \in \Lambda} \quad \text{with multiplicities}$$



$$\text{RHS} = \frac{\operatorname{Vol} \Pi_\Lambda}{4\pi t^{n/2}} \cdot \sum_{\ell \in \tilde{L}} e^{-\frac{\ell^2}{4t}}$$

multiplicity

$$\text{LHS} = \text{RHS}$$

Spectral Geometrical

$\sum_{\lambda \in Sp_\Delta(\Pi_\Lambda)} e^{-\lambda t}$ has a "big" singularity at $t \to 0^+$

At the first order we have:

$$\lim_{t \to 0^+} \text{LHS} = \frac{\operatorname{Vol} \Pi_\Lambda}{4\pi t^{n/2}}$$

So! I can recover n!

$$\lim_{t \to 0^+} \frac{\log \text{LHS}}{\log t} = \cancel{\frac{\log \left(\frac{\operatorname{Vol} \Pi_\Lambda}{4\pi} \right)}{\log t}}^{\,0} - \frac{n}{2}$$

so $n = \lim_{t \to 0^+} -2 \frac{\log \text{LHS}}{\log t} = n$

$\to$ n is Δ-spectrally determined

Moreover: $\lim_{t \to 0^+} 4\pi t^{n/2}\, LHS = \mathrm{Vol}\, \mathbb{T}_\Lambda$

so: $\mathrm{Vol}\, \mathbb{T}_\Lambda$ is Δ-spectrally determined.

- Moreover:

$$\lim_{t \to 0^+} \boxed{\frac{4\pi t^{n/2}\, LHS}{\mathrm{Vol}\, \mathbb{T}_\Lambda} - 1} \; (*) = \sum_{\ell \in \widetilde{LS} \setminus \{0\}} e^{-\frac{\ell^2}{4t}}$$

if $\ell > \ell'$, $e^{-\ell/4t}$ goes to 0 faster (as $t \to 0^+$) then $e^{-\ell'/4t}$

so let $\underline{\ell} = \min \widetilde{LS} \setminus \{0\}$ = minimal length of a closed geodesics in $\mathbb{T}_\Lambda$

multiplicity of $\underline{\ell}$

$m_{\underline{\ell}} \cdot e^{-\frac{\underline{\ell}^2}{4t}}$ is the dominant term in $(*)$

so $4t \cdot \log (*) \simeq \cancel{4t \log m_{\underline{\ell}}} - \underline{\ell}^2$

$\xrightarrow[t \to 0^+]{} -\underline{\ell}^2$

$t \to 0$

$(*) \cdot e^{\frac{\underline{\ell}^2}{4t}} \longrightarrow m_{\underline{\ell}}$

so $\underline{\ell}$ is Δ-spec. determined

$m_{\underline{\ell}}$ is ——— " ———.

If we keep removing larger asymptotics, we can find, by induction, all lengths of geodesics + multiplicities.

Rmk: This is very fragile since ℓ's are determined by delicate asymptotics

$\Rightarrow$ not associated to singularities $\rightsquigarrow$ easily broken by perturbations.

This describes the direction $\Delta \longrightarrow$ Length.

What about Length $\longrightarrow \Delta$?

$$\sum_{\lambda \in Sp_\Delta(\mathbb{T}_\Lambda)} e^{-\lambda t} = \frac{\mathrm{Vol}\ \mathbb{T}_\Lambda}{4\pi t^{n/2}} \sum_{\ell \in \widetilde{LS}(\mathbb{T}_\Lambda)} e^{-\frac{\ell^2}{4t}}$$

Q: Can I recover Vol $\mathbb{T}_\Lambda$, n by LS?

In general: $\#\{\text{geodesics of length} < L\}$

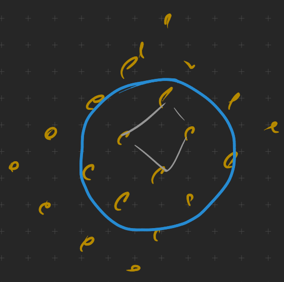
$$\lesssim e^{h_{top} L}$$ $\leftarrow$ TOPOLOGICAL ENTROPY of Geod. flow.

Luckily, on $\mathbb{T}_\Lambda$ $h_{top} = 0$ and the growth is polynomial. In fact

$$LS(\mathbb{T}_\Lambda) = \{ \|s\| \mid s \in \Lambda \} \quad \text{so}$$

$\#\{\text{geodesics of length} \leq L\} \approx$

$\#$ lattice pts in a ball of radius L



$$\approx \frac{B_n \cdot L^n}{\mathrm{vol}\, \mathbb{T}_\Lambda} + o(L^n)$$

n: LS-determined

$\mathrm{vol}\, \mathbb{T}_\Lambda$: LS-determined

B_n: volume of unit n-ball

For the torus (only!) LS $\rightsquigarrow$ $\mathrm{vol}\, \mathbb{T}_\Lambda$

$\rightsquigarrow$ dimension.

The wave trace.

$$\sum_{\lambda \in Sp_\Delta(\lambda)} e^{i\sqrt{\lambda}\, t} \approx \mathrm{Tr}\left(e^{-i\sqrt{\Delta}\, t}\right)$$

Naively, we would like to apply PSF to

$$f(\xi) = e^{i 2\pi \|\xi\| t}$$

then I would have

$$\sum_{\xi \in \Lambda^*} e^{i 2\pi \|\xi\| t}$$

This is problematic since

$$e^{i 2\pi \|\xi\| t} \notin S.$$

Step 1. Write PSF for radial Schwartz functions

Def: $f:\mathbb{R}^n \to \mathbb{C}$ is radial if $\exists\, \varphi : \mathbb{R}_{\geq 0} \to \mathbb{C}$ s.t. $f(x) = \varphi(\|x\|)$

Fact: if $f \in S(\mathbb{R}^n)$, f radial, then $f(x) = \varphi(\|x\|)$ for $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{C}$;

φ can be extended to a C^∞ even function: $\mathbb{R} \to \mathbb{C}$. (Abusing notation, let us call the extension also φ).

$$\varphi \in S(\mathbb{R}).$$

Apply PSF to a radial f:

$$\sum_{\xi \in \Lambda^*} f(\xi) = \operatorname{covol} \Pi_\Lambda \cdot \sum_{s \in \Lambda} \hat{f}(s)$$

$$\|$$

$$\sum_{\xi \in \Lambda^*} \varphi(\|\xi\|) = \operatorname{covol} \Pi_\Lambda \cdot \sum_{s \in \Lambda} \tilde{\varphi}(\|s\|)$$

where $\tilde{\varphi}$ is an integral transform of φ.

$$\|$$

$$\left[\sum_{\xi \in \Lambda^*} \delta_{\|\xi\|}\right] \varphi$$

$$\delta_y(\varphi) := \varphi(y)$$ linear functional on C^0

$\delta_y \in S'$

tempered distribution.

$$\left[\sum_{\xi \in \Lambda^*} \delta_{\|\xi\|}\right]$$ defines a functional on S

an element of S'

tempered distribution

" Dirac comb "

interpret PSF as an equation on tempered distributions.

$$\circledast \quad \sum_{\xi \in \Lambda^*} \delta_{\|\xi\|} = \operatorname{covol} \mathbb{T}_V \sum_{s \in \Lambda} \Phi_s$$

evolution of the int. transform of s

Recall One can take Fourier transform of

$\circledast$:

Recall $$\left[\check{\mathcal{F}} \delta_y\right](t) = \int e^{-2\pi i \omega t} \delta_y(\omega)\, d\omega$$

$$= e^{-2\pi i y t}, \text{ so}$$

$$\check{\mathcal{F}}(\mathrm{LHS}) = \sum_{\xi \in \Lambda^*} e^{-2\pi i \|\xi\| t}$$

wave trace of Δ.

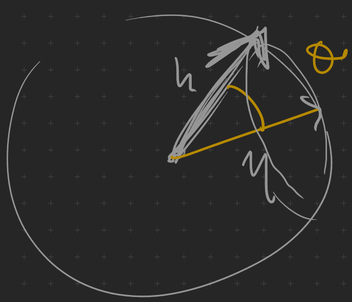
Figure out what is $\check{\varphi}$; let $\mathbb{R}^n \ni s = \sigma \cdot h$, $\sigma \geq 0$, $\|h\| = 1$

$$\check{f}(s) = \int_{\mathbb{R}^n} e^{2\pi i \langle \xi, s \rangle} f(\xi)\, d\,\mathrm{Vol}(\xi)$$

$$= \int_{\mathbb{R}^n} e^{2\pi i \langle \xi, s \rangle} \varphi(\|\xi\|)\, d\,\mathrm{Vol}(\xi)$$

$$\check{f}(\sigma h) = \int_{\mathbb{R}^n} e^{2\pi i \sigma \langle \xi, h \rangle} \varphi(\|\xi\|)\, d\,\mathrm{Vol}(\xi)$$

take polar coord. s.t. axis 1 is in the direction h



$$= \int_{\mathbb{R}_{\geq 0}} r^{n-1}\, dr \int_{S^{n-1}} d(S^{n-1})[\eta]\; e^{2\pi i \sigma r \langle \eta, h \rangle} \varphi(r)$$

$$= \int_0^\infty \varphi(r)\, r^{n-1} \underbrace{\int_{S^{n-1}} e^{2\pi i \sigma r \cos\theta}\, dS^{n-1}[\eta]}_{\substack{\| \\ K_{n-2}(\sigma r)}}$$

Where

$$K_m(\lambda) = \int_{-\pi}^{\pi}\int_0^{\pi}\cdots\int_0^{\pi} e^{2\pi i \lambda \cos\vartheta} \sin^m \vartheta \, \sin^{m-1}\vartheta_2 \cdots \sin\vartheta_m \, d\vartheta_1 \cdots d\vartheta_m \, d\vartheta_{m+1}$$

$$= \omega_m \int_0^{\pi} e^{2\pi i \lambda \cos\vartheta} \sin^m \vartheta \, d\vartheta$$

vol of S^m

Def $$\tilde{\varphi}(\sigma) = \int_0^{\infty} \varphi(r)\, K_{n-2}(\sigma r)\, r^{n-1}\, dr$$