Λ lattice in $\mathbb{R}^n$, $\Lambda^* = \{\xi \in \mathbb{R}^n \mid \langle \xi, x \rangle \in \mathbb{Z}\}$

Dual lattice to Λ.

Let $D_i = \frac{1}{2\pi i} \frac{\partial}{\partial x_i}$ $\hat{D}_i = \frac{i}{2\pi} \frac{\partial}{\partial \xi_i}$

Schwartz functions: $f \in S(\mathbb{R}^n; \mathbb{C})$ if $f \in C^\infty(\mathbb{R}^n; \mathbb{C})$

and $\| x^\alpha D^\beta f \|_\infty < \infty$

$\forall$ α, β multi-index

Recall $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ $\beta \in \mathbb{N}^n$

$x^\alpha = \prod x_i^{\alpha_i}$ $D^\beta = D_n^{\beta_n} \cdots D_2^{\beta_2} D_1^{\beta_1}$

Example:

- $C_0^\infty(\mathbb{R}^n; \mathbb{C}) \subset S(\mathbb{R}^n; \mathbb{C})$ (compactly supported C^∞ functions)
- $\frac{1}{1 + \|x\|^2} \notin S$
- $P(x) e^{-\|x\|^2} \in S(\mathbb{R}^n; \mathbb{C})$

↳ polynomial

Fourier transform: Let $f \in S(\mathbb{R}^n; \mathbb{C}) =: S(\mathbb{R}^n)$

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i \langle \xi, x \rangle} f(x)\, dx$$

$$\check{f}(x) = \int_{\mathbb{R}^n} e^{2\pi i \langle \xi, x \rangle} f(\xi)\, d\xi$$

Note: 1. $\hat{f}$ is well-defined (I just need $f \in L^1(\mathbb{R}^n)$ for $\int$ to make sense)

$\forall f \in S$

Facts: - $\widehat{D^\alpha f}(\xi) = \xi^\alpha \hat{f}(\xi)$

- $\widehat{x^\alpha f}(\xi) = \hat{D}^\alpha \hat{f}(\xi)$

$\forall f, g \in S$

Fubini $\Rightarrow$ $\int \hat{f} g(x)\, dx = \int f(\xi) \hat{g}(\xi)\, d\xi$

Cor if $f \in S$, then $\hat{f} \in S$

Let $\mathcal{F} : f \longmapsto \hat{f}$ is a continuous bijection from $S \longrightarrow S$

in fact. $\mathcal{F}^{-1} : f \longmapsto \check{f}$

Poisson's Summation Formula

span $\Lambda = \mathbb{R}^n$, $\mathbb{Z}^n \xrightarrow{\text{iso}} \Lambda$

$\forall f \in S(\mathbb{R}^n)$, $\forall \Lambda$ full-rank lattice

$$\sum_{\xi \in \Lambda^*} f(\xi) = \operatorname{covol}\Lambda \cdot \sum_{x \in \Lambda} \check{f}(x)$$

Pf: Let $\Phi(\eta) := \sum_{\xi \in \Lambda^*} f(\eta + \xi)$ (*)

Claim: i) Φ is well defined, ii) Φ is Λ^*-periodic

Pf: ii) is trivial: Assure (*) converges,

then $\forall \bar{\xi} \in \Lambda^*$ $\Phi(\eta + \bar{\xi}) = \sum_{\xi \in \Lambda^*} f(\eta + \bar{\xi} + \xi)$

$$= \sum_{\xi' \in \Lambda^*} f(\eta + \xi') = \Phi(\eta)$$

i) implied by $f \in S(\mathbb{R}^n)$: let us

define $\Phi_N(\eta) = \sum_{\substack{\xi \in \Lambda^* \\ \|\xi\| \le N}} f(\eta + \xi)$

by ii) it suffices to study $\eta \in D_{\Lambda^*}$

$\lim_{N\to\infty} \phi_N = \phi$ unif'ly.

$$\| \phi_{N+1} - \phi_N \|_\infty \longrightarrow 0 \text{ with } N$$

$$\| \phi_{N+1} - \phi_N \| = \Big\| \sum_{\substack{\xi \in \Lambda^* \\ N < \|\xi\| \le N+1}} \hat{f}(\eta + \xi) \Big\| \quad (*)$$

there are at most $\#\{ \xi \in \Lambda^* : \|\xi\| \le N+1 \}$

$$\le 2 \frac{\mathrm{Vol}\left(B_{\mathbb{R}^n}(N+1) \right)}{\mathrm{covol}\ \Lambda} \sim N^n$$

since $f \in S$ if $N \gg \|\eta\|$, then

$$\hat{f}(\eta + \xi) \le C / N^{10n}$$

$$(*) \le C N^{-9n} \text{ summable}$$

apply to any $D^\alpha f$, too.

so ϕ is C^∞, Λ^*-periodic. $\square$

What can I do with a Λ^* periodic C^∞ function?

$$\phi(\eta) = \sum_{s \in \Lambda} c_s e^{2\pi i \langle s, \eta \rangle}$$

where $$c_s = \frac{1}{\operatorname{covol} \Lambda^*} \int_{\Pi_\Lambda} \phi(\eta) e^{-2\pi i \langle s, \eta \rangle} \, d\eta$$

$$= \frac{1}{\operatorname{covol} \Lambda^*} \int_{\Pi_{\Lambda^*}} \sum_{\xi \in \Lambda^*} f(\eta + \xi) e^{-2\pi i \langle s, \eta \rangle} \, d\eta$$

BCT

$$= \frac{1}{\operatorname{covol} \Lambda^*} \sum_{\xi \in \Lambda^*} \int_{D_{\Lambda^*}} f(\eta + \xi) e^{-2\pi i \langle s, \eta \rangle} \, d\eta$$

$$= \frac{1}{\operatorname{covol} \Lambda^*} \int_{\mathbb{R}^n} f(\eta) e^{-2\pi i \langle s, \eta \rangle} \, d\eta$$

$$c_s = \frac{1}{\operatorname{covol} \Lambda^*} \check{f}(s)$$

$$\check{\phi}(\eta) = \operatorname{covol} \Lambda \sum_{s \in \Lambda} \check{f}(s) e^{2\pi i \langle s, \eta \rangle}$$

compute $\uparrow$ in $\eta = 0$

$$\sum_{s \in \Lambda^*} f(s) = \phi(0) = \text{covol}\,\Lambda \sum_{s \in \Lambda} \hat{f}(s)$$

$$\eta \left\{ \sum_{\xi \in \Lambda^*} \delta_\xi \right\} f$$

□

we will use $\uparrow$ to compute Heat trace and Wave trace on T_Λ

$P : \mathcal{H}$ that admits a orthonormal basis of e-vectors.

$$\text{Tr}(P) = \sum_i \lambda_i$$ ← if conv's obs

we say P is of trace class.

$$U(t) = e^{-\Delta \cdot t}$$

$$\text{Tr}(U(t)) =$$

$P = D_i$ $\quad e^{-iD_i t} = U(t)$

generates translations by t

$\mathrm{Tr}\left(e^{-it\frac{d}{dx}}\right)$ computed using Poisson's summation.