

The study of dispersive equations has emerged in the past decade as an accessible venue to refine new techniques for nonlinear PDE. The cubic focusing nonlinear Schrödinger equation, $iu_t + \Delta u + u|u|^2 = 0$, demonstrates a mixture of standard and exotic properties, and so makes an excellent subject for study. There exist Strichartz estimates to establish local-wellposedness, the nonlinearity is always smooth, the linear Schrödinger operator is very well understood, there is some understanding of standing-wave solutions, and there is some physical motivation (especially in dimensions two and three). Nevertheless, the nature of the equation can change dramatically with dimension, solutions can display infinite speed of propagation, and solutions can blowup spectacularly in finite time.

Properties of solutions that blowup have been studied intensively in two-dimensions. The analysis of 2D cubic NLS is helped by its unique scaling properties, namely that the L^2 norm invariant under spatial scaling is also a conserved quantity. Combined with a variational characterization of the ground-state standing-wave solution, this has led to Merle and Raphaël's comprehensive proof of so-called log-log behaviour, solutions to 2D cubic NLS that blowup according to,

$$\|u(t)\|_{H^1} \sim \sqrt{\frac{\log |\log(T-t)|}{T-t}}.$$

With an assumption of cylindrical symmetry, we may exploit our knowledge of the two-dimensional problem to demonstrate new behaviour in the three-dimensional problem.

Theorem (Z. 2009). *There is data in $H^1(\mathbb{R}^3)$ for 3D cubic NLS that leads to blowup precisely on a circle in finite time. This blowup occurs in the sense of:*

- **Mass Concentration.** *The modulus square of the central profile of the solution converges to a delta function on a circle with fixed positive radius. The remainder of the solution has a strong limit in L^2 at blowup time.*
- **Regularity Away from Singular Set.** *For any $R > 0$, the solution is bounded in H^1 norm outside the toroid of radius R around the blowup circle.*

The method of proof also establishes the existence of data in $H^1(\mathbb{R}^N)$ that blowup on a set of co-dimension two under the N -dimensional cubic NLS evolution.

Idea of Proof.

In cylindrical coordinates $x = (r, z, \theta)$ we write the Laplacian,

$$\Delta_x = \partial_r^2 + \partial_z^2 + \frac{\partial_r}{r}.$$

Let us rescale a cylindrically symmetric solution $u(t, x)$ around the circle $\{(1, 0, \theta) : 0 < \theta \leq 2\pi\}$,

$$u(t, x) = \frac{1}{\lambda} v \left(\frac{t}{\lambda^2}, \frac{(r, z) - (1, 0)}{\lambda} \right).$$

The function v is supported on a half-plane of \mathbb{R}^2 . After changing coordinates, v satisfies,

$$i\partial_s v + \Delta_y v + \frac{\lambda}{r} \partial_{y_1} v + v|v|^2 = 0 \quad \text{where } s = \frac{t}{\lambda^2}, \quad y = \frac{(r, z) - (1, 0)}{\lambda}.$$

Consider a solution $u(t, x)$ that is tightly concentrated on a toroid of width $\lambda \ll 1$ around the circle $\{(1, 0, \theta) : 0 < \theta \leq 2\pi\}$. The profile of $u(t, x)$ near the circle is then renormalized to have support

of fixed width in the new variable y . The evolution of v is then essentially that of 2D cubic NLS with a perturbation of the order $\frac{\lambda}{r} \sim \frac{\lambda}{1} \ll 1$. As blowup proceeds, the solution $u(t, x)$ concentrates further, λ grows smaller still, and the perturbation $\frac{\lambda}{r} \partial_{y_i} v$ becomes less relevant. The dynamics of v progressively approach those of 2D cubic NLS as blowup occurs.

Similar Work.

Raphaël was the first to demonstrate blowup of nonlinear Schrödinger at more than a simple collection of points, by showing a class of radially symmetric data that blowup on a ring under two-dimensional quintic NLS. This work was later extended by Raphaël and Szeftel to produce blowup on spheres for all quintic problems, including the energy supercritical case.

Strategy of Proof.

The proof proceeds in three steps.

1. Control the growth of a higher-order norm in terms of the H^1 norm. We start with regular data, and exploit that the solution can be decomposed into a dominant central profile, which is smooth, and an error term. Although in theory the error term may possess the larger higher-order norm, it is of significantly lower order in H^1 , which allows our estimate to carry through.
2. Prove unusual regularity persists on a toroidal shell surrounding the singular ring, insulating the truly three-dimensional region from the singular dynamics. This is the most difficult part, due to the failure of Sobolev embedding $H^1(\mathbb{R}^2) \not\hookrightarrow L^\infty(\mathbb{R}^2)$.
3. Use integrability of the log-log rate in time to upgrade the improvement from step 2, in stages. This idea was introduced by Raphaël and substantially refined by Raphaël and Szeftel. The final product allows us to guarantee the two-dimensional dynamic on the ring is unaffected by activity elsewhere.

Blowup of 3D Cubic NLS.

Our examples demonstrate new geometry and speed of blowup. Asymptotics and numerics of Fibich, Gavish and Wang have suggested there exist radially symmetric blowup solutions that collapse to a point. The H^1 norm of these collapsing sphere solutions are believed to grow at the order of $(T - t)^{-\frac{1}{3}}$. Self-similar solutions, that blowup at a point with H^1 growth rate $(T - t)^{-\frac{1}{4}}$, have been observed numerically by McLaughlin, Papanicolaou, Sulem and Sulem. Our examples are much faster than either scenario.

More generally, the classic virial identity gives a sufficient condition for data to blowup under 3D cubic NLS, and Holmer and Roudenko have proven a sharp criteria for non-blowup in some cases. Our examples include data that demonstrate blowup is possible outside the scope of existing blowup theory. Furthermore, the L^3 norm of our examples grow explicitly of the order $(T - t)^{-\frac{1}{6}}$, which answers an open question regarding a theorem of Merle and Raphaël.