Last Name:____________________ First Name:____________________

MAT 244S, GHA # 2. Feb. 10-25, 98
Front page

Read Me First! (general rules for GHA)

- Don't write your student ID on this page!
- Take care to staple your homework! (no clips, please)
- Be nice, write neatly, use pen, please!
- This GHA constitutes 5% of the final mark (1 pt = 1%)
- You must deliver it to me until 21.30 pm Jan 27 in my office or earlier (on the lecture)
- Don't leave in mailboxes!

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1) *0.8 pt* Find (a) The general solution to

$$y'' - 3y' - 4y = 48e^x \cos x + 6e^x - 5e^{-x}$$

and (b) Solution satisfying $y(0) = y'(0) = 0$.

2) *0.8 pt* Find (a) The general solution to

$$y'' - 6y' + 10y = e^{3x} + 39\cos x + 20e^{3x}\cos x$$

and (b) Solution satisfying $y(0) = y'(0) = 0$.

3) *1pt* Find (a) The general solution to

$$y'' - y' - 2y = \frac{e^{3x}}{e^x + 1}$$

and (b) Solution satisfying $y(0) = y'(0) = 0$.

4) *1 pt* For equation

$$y'' - \frac{y'}{\cos x \sin x} - y\tan^2 x = 0$$

(a) derive equation for Wronskian and resolve it;
(b) check that $y_1 = \cos x$ is a solution and find another non-proportional solution;
(c) Find Wronskian of these two solutions;

5) *0.8 pt* Find (a) The general solution to

$$y'' = \frac{y'}{y^2}$$

and (b) Solution satisfying $y(0) = 1$, $y'(0) = -1$.

6) *0.8 pt* Find (a) The general solution to

$$y'' - \tanh x y' = \cosh x$$

and (b) Solution satisfying $y(0) = y'(0) = 0$.

7) *0.8 pt* Find (a) The general solution to

$$y''' - 4y'' + 5y' - 2y = e^x + e^{2x} + 10\cos x$$

and (b) Solution satisfying $y(0) = y'(0) = y''(0) = 0$.

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Back page

FOR MARKER'S USE ONLY	
Question	Mark
1	/0.8
2	/0.8
3	/1.0
4	/1.0
5	/0.8
6	/0.8
7	/0.8
TOTAL	/5.0

That's right, you can get 6 points out of 5!

Solutions

1) Finding characteristic roots:

$$\lambda^2 - 3\lambda - 4 = 0 \implies \lambda_1 = -1, \lambda_2 = 4$$

and the solution of homogeneous equation is

$$y^* = C_1 e^{-x} + C_2 e^{4x}.$$

Equation with r.h.e. $e^x \cos x$ has a solution of the form $\bar{y}_1 = e^x(A\cos x + B\sin x)$ because $1 \pm i$ are not characteristic roots. Then $\bar{y}_1' = e^x\big((A+B)\cos x + (B-A)\sin x\big)$, $y_1'' = 2e^x(-A\sin x + B\cos x)$ and plugging into equation we get

$$e^x\big((B-7A)\cos x + (A-7B)\sin x\big) = 48e^x\cos x \implies B = -1, A = 7 \implies \bar{y}_1 = 7e^x\cos x - e^x \sin x.$$

Equation with r.h.e. $6e^x$ has a solution of the form $\bar{y}_2 = Ae^x$; plugging into equation we get $A = -1$ and $\bar{y}_2 = -e^x$.

Equation with r.h.e. $-5e^{-x}$ has a solution of the form $\bar{y}_3 = Axe^{-x}$ because $\lambda = -1$ is a simple characteristic root; plugging into equation we get $A = -1$, $\bar{y}_3 = -xe^{-x}$.

Now

$$y = 7e^x\cos x - e^x\sin x - e^x + xe^{-x} + C_1e^{-x} + C_2e^{4x}$$

solves (a); initial conditions of (b) are satisfied as $C_1 = -\frac{18}{5}, C_2 = -\frac{12}{5}$.

2) Finding characteristic roots:

$$\lambda^2 - 6\lambda + 10 = 0 \implies \lambda_{1,2} = 3 \pm i$$

and the solution of homogeneous equation is

$$y^* = e^{3x}(C_1\cos x + C_2\sin x).$$

Equation with r.h.e. e^{3x} has a solution of the form $\bar{y}_1 = Ae^{3x}$ because 3 is not a characteristic root. Plugging into equation we get $A = 1$ and $\bar{y}_1 = e^{3x}$.

Equation with r.h.e. $39\cos x$ has a solution of the form $\bar{y}_2 = A\cos x + B\sin x$; plugging into equation we get $A = 3$, $B = -2$ and $\bar{y}_2 = 3\cos x - 2\sin x$.

Equation with r.h.e. $20e^{3x}\cos x$ has a solution of the form $\bar{y}_3 = xe^{3x}\big(A\cos x + B\sin x\big)$ because $\lambda = 3 \pm i$ are simple characteristic roots; plugging into equation we get

$$e^x\big((2B-4A)\cos x - (2A+4B)\sin x\big) = 20e^x\cos x$$

and $A = -4$, $B = 2$, $\bar{y}_3 = xe^{3x}(-4\cos x + 2\sin x)$.

Now

$$y = e^{3x} + 3\cos x - 2\sin x + xe^{3x}(-4\cos x + 2\sin x) + e^{3x}(C_1\cos x + C_2\sin x)$$

solves (a); initial conditions of (b) are satisfied as $C_1 = -4, C_2 = 17$.

3) Finding characteristic roots $\lambda_1 = -1, \lambda_2 = 2$ we have solution of the homogeneous equation

$$y = C_1 e^{-x} + C_2 e^{x}$$

and applying variation of constants we arrive to

$$\begin{cases} C_1' e^{-x} + \ C_2' e^{2x} = 0, \\ -C_1' e^{-x} + 2C_2' e^{2x} = \dfrac{e^{3x}}{e^x + 1}; \end{cases}$$

solving this system we get

$$\begin{aligned} C_1' &= -\frac{1}{3}\frac{e^{4x}}{e^x+1} = -\frac{1}{3}\big(e^{3x} - e^{2x} + e^x\big) + \frac{1}{3}\frac{e^x}{e^x+1}, \\ C_2' &= \frac{1}{3}\frac{e^x}{e^x+1}; \end{aligned}$$

then

$$\begin{aligned} C_1 &= -\frac{1}{3}\big(\frac{1}{3}e^{3x} - \frac{1}{2}e^{2x} + e^x\big) + \frac{1}{3}\log(e^x+1) + \bar{C}_1, \\ C_2 &= \frac{1}{3}\log(e^x+1) + \bar{C}_2 \end{aligned}$$

with constant $\bar{C}_1, \bar{C}_2$ and

$$y = -\frac{1}{3}\big(\frac{1}{3}e^{2x} - \frac{1}{2}e^x + 1\big) + \frac{1}{3}^{-x}\log(e^x+1) + \bar{C}_1 e^{-x} + \frac{1}{3}e^{2x}\log(e^x+1) + \bar{C}_2 e^{2x}$$

solves (a); initial conditions of (b) are satisfied as $\bar{C}_1 = \frac{5}{18} - \frac{1}{3}\log 2, \bar{C}_2 = -\frac{1}{3}\log 2$.

4)

(a) According to theory

$$W' - \frac{W}{\cos x \sin x} = 0 \implies \frac{dW}{W} = \frac{2dx}{\sin 2x} \implies$$
$$\log W = \log\tan x + \log C \implies W = C\tan x.$$

(b) Substituting $y_2 = z\cos x,\ z' = u$ into equation we get

$$z''\cos x + z'(-2\sin x - \frac{1}{\sin x}) = 0 \implies \frac{du}{u} = (2\tan x + \frac{2}{\sin 2x})dx \implies$$
$$\log u = (-2\log\cos x + 2\log\tan x)$$

where we picked arbitrarily a constant after integration; then

$$u = z' = \frac{\sin x}{\cos^3 x} \implies z = \frac{1}{\cos^2 x}$$

where we picked the constant again; finally $y_2 = \frac{1}{\cos x}$.

(c) Then

$$W = \begin{vmatrix} \cos x & \frac{1}{\cos x} \\ -\sin x & \frac{\sin x}{\cos^2 x} \end{vmatrix} = -\tan x$$

in correspondence with (a).

Remark. Using (a) one could write equation for $y = y_2$:

$$W = \begin{vmatrix} \cos x & y \\ -\sin x & y' \end{vmatrix} = y' \cos x + y \sin x = \tan x;$$

solving this equation one gets $y = -\frac{1}{\cos x}$ (after we pick up a constant).

5) Let $z = y'$, then

$$z\frac{dz}{dy} = \frac{z}{y^2} \implies dz = \frac{dy}{y^2} \implies z = -\frac{1}{y} + C$$

in the general case; taking $y = 1$, $z = -1$ we get $C = 0$ and $z = -\frac{1}{y}$ in the case (b).

Then

$$dx = \frac{dy}{C - \frac{1}{y}} = C_1 \frac{ydy}{y - C_1} = C_1 dy + C_1^2 \frac{dy}{y - C_1}$$

with $C_1 = \frac{1}{C}$ and

$$x = C_1 y + C_1^2 \log(y - C_1) + C_2$$

solves (a). In the case (b)

$$\frac{dy}{dx} = -\frac{1}{y} \implies dx = -ydy \implies x = -\frac{y^2}{2} + C_2$$

with $C_2 = \frac{1}{2}$ due to $y(0) = 1$. Then $y = \sqrt{1 - 2x}$ solves (b).

6) Let $z = y'$; then

$$z' - \tanh xz = \cosh x$$

is a first order linear equation; solving homogeneous equation first

$$\frac{dz}{dx} = z \tanh x \implies \frac{dz}{z} = \tanh x dx \implies \log z = \log \cosh x + \log C \implies z = C \cosh x$$

and assuming that C is a function and plugging $z = C \cosh x$ into equation we get

$$C' = 1 \implies C = x + C_1 \implies z = (x + C_1) \cosh x$$

and

$$y = \int (x \cosh x + C_1 \cosh x) dx = x \sinh x - \cosh x + C_1 \sinh x + C_2$$

with constants $C_{1,2}$ solves (a) and $y = x\sinh x - \cosh x + 1$ solves (b).
7) Finding characteristic roots:

$$\lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0$$

and one can guess $\lambda_1 = 1$ and $\lambda_3 = 2$ and then $\lambda_2 = 1$ (Vieta theorem). Now

$$y^* = C_1 e^x + C_2 x e^x + C_3 e^{2x}$$

solves homogeneous equation.

Equation with r.h.e. e^x has a solution $\bar{y}_1 = Ax^2 e^x$ because $\lambda = 1$ is a double root. Plugging into equation we get $A = -\frac{1}{2}$, $\bar{y}_1 = -\frac{1}{2}x^2 e^x$.

Equation with r.h.e. e^{2x} has solution $\bar{y}_2 = Axe^{2x}$ because $\lambda = 2$ is a simple root. Plugging into equation we get $A = 1$, $\bar{y}_2 = xe^{2x}$.

Equation with r.h.e. $10\cos x$ has a solution $\bar{y}_3 = A\cos x + B\sin x$. Plugging into equation we get $2A + 4B = 10, -4A + 2B = 0 \implies A = 1, B = 2$ and $\bar{y}_3 = \cos x + 2\frac{1}{5}\sin x$.

Finally,

$$-\frac{1}{2}x^2 e^x + xe^{2x} + \cos x + 2\sin x + C_1 e^x + C_2 x e^x + C_3 e^{2x}$$

solves (a); initial conditions of (b) are satisfied as $C_1 = -4, C_2 = 0, C_3 = 3$.