

Last Name: _____ First Name: _____

MAT 244S, GHA # 1. Jan. 20-27, 98

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Read Me First! (general rules for GHA)

- Don't write your student ID on this page!
 - Take care to staple your homework! (no clips, please)
 - Be nice, write neatly, use pen, please!
 - This GHA constitutes 5% of the final mark (1 pt = 1%)
 - You must deliver it to me until 21.30 pm Jan 27 in my office or earlier (on the lecture)
 - Don't leave in mailboxes!
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1) *1pt* Find (a) The general solution to

$$\frac{dy}{dx} = \frac{1 + y^2}{1 + x^2}$$

and (b) Solution satisfying $y(0) = 1$.

2) *1pt 1) 1pt* Find (a) The general solution to

$$\frac{dy}{dx} - xy = xe^{-\frac{x^2}{2}}$$

and (b) Solution satisfying $y(0) = 0$.

3) *1pt* Find (a) The general solution to

$$\frac{dy}{dx} - y \tan x = \frac{1}{y^2 \cos^2 x}$$

and (b) Solution satisfying $y(0) = 1$.

4) *1pt* Find (a) The general solution to

$$\frac{dy}{dx} = \frac{2xy + y^2}{x^2 + xy}$$

and (b) Solution satisfying $y(1) = 1$.

5) *1pt* Check that this equation

$$(xy + 1)e^{xy} dx + x^2 e^{xy} dy = 0$$

is exact and find (a) The general solution to

and (b) Solution satisfying $y(1) = 0$.

6) (bonus) *1pt* Find integrating factor for

$$y dx + (2xy - e^{-2y}) dy = 0$$

and find the general solution.

Solutions

1) Separating variables:

$$\frac{dy}{1+y^2} = \frac{dx}{1+x^2}$$

and integrating

$$(1) \quad \arctan y = \arctan x + C$$

implies the *general* solution

$$y = \tan(\arctan x + C) = \frac{x+a}{1-ax}$$

where $a = \tan C$ and we used that $\tan \arctan x = x$ and $\tan(\alpha + \beta)$ formula.

Further, (1) and initial condition yield $C = \frac{\pi}{4}$, $a = 1$ and $y = \frac{1+x}{1-x}$ is the *solution to initial problem*.

2) This is linear equation; solving homogeneous equation

$$y' - xy = 0 \implies \frac{dy}{y} = x dx \implies \log y = \frac{x^2}{2} + \log C \implies y = C e^{\frac{x^2}{2}}$$

and plugging $y = z e^{\frac{x^2}{2}}$ into original equation we get

$$\frac{dz}{dx} e^{\frac{x^2}{2}} = x e^{-\frac{x^2}{2}} \implies z = \int x e^{-x^2} dx = -\frac{1}{2} e^{-x^2} + C \implies y = -\frac{1}{2} e^{-\frac{x^2}{2}} + C e^{\frac{x^2}{2}}$$

is the general solution.

Using $y(0) = 0$ we find $C = \frac{1}{2}$ and $y = -\frac{1}{2} e^{-\frac{x^2}{2}} + \frac{1}{2} e^{\frac{x^2}{2}}$ is the solution to (b).

3) This is Bernoulli equation; solving linear homogeneous equation

$$\begin{aligned} \frac{dy}{dx} - y \tan x = 0 &\implies \frac{dy}{y} = \tan x dx \implies \\ \log y = \int \tan x dx &= -\log \cos x + \log C \implies y = \frac{C}{\cos x} \end{aligned}$$

and plugging $y = \frac{z}{\cos x}$ we get

$$\frac{dz}{dx} (\cos x)^{-1} = z^{-2} \implies z^2 dz = \cos x dx \implies \frac{z^3}{3} = \sin x + \frac{C}{3} \implies z = (3 \sin x + C)^{\frac{1}{3}}$$

which yields the general solution $y = (3 \sin x + C)^{\frac{1}{3}} (\cos x)^{-1}$.

Then $y(0) = 1$ yields $C = 1$ and $y = (3 \sin x + 1)^{\frac{1}{3}} (\cos x)^{-1}$ solves (b).

4) This is homogeneous equation; picking $y = xz$ we get

$$x \frac{dz}{dx} + z = \frac{2z + z^2}{1 + z} \implies x \frac{dz}{dx} = \frac{z}{1 + z} \implies \left(1 + \frac{1}{z}\right) dz = \frac{dx}{x} \implies z + \log z = \log x + C$$

which yields that $\frac{y}{x} + \log y = 2 \log x + C$ is the general solution.

Taking $x = 1, y = 1$ we get $C = 0$ and $\frac{y}{x} + \log y = 2 \log x$ is solution to (b).

5) This is exact equation:

$$\frac{\partial}{\partial y} ((xy + 1)e^{xy}) = (x^2y + 2x)e^{xy} = \frac{\partial}{\partial x} (x^2e^{xy}).$$

Note, that $(xy + 1)e^{xy} = 1$ as $y = 0$. Then taking

$$U(x, y) = \int_0^x 1 dx + \int_0^y x^2 e^{xy} dy$$

with independent x, y we get

$$U(x, y) = x + xe^{xy} \Big|_{y=0}^{y=y} = xe^{xy}$$

which gives us the general solution $xe^{xy} = C$.

Taking $x = 1, y = 0$ we get $C = 1$ and $xe^{xy} = 1$ is a solution to (b).

6) This is not exact equation (check by yourself!) But $dx + 2xdy$ “needs” factor e^{2y} to become $d(xe^{2y})$ and so $y^{-1}e^{2y}$ looks as a factor in question. Multiplying by it we get

$$d(xe^{2y}) - \frac{dy}{y} = 0 \implies xe^{2y} + \log y = C$$

as a general solution.