MAT 244S, Final Exam, May 5, 1998. Problems

FE-1 (*4 pts*) Solve

$$y'' + y'\frac{2x}{x^2+1} = x(1+x^2)^{-\frac{3}{2}}$$

FE-2 (*4 pts*) Find solution to

$$y'' = 2yy'.$$

FE-3 (*5.5 pts*) Check that $y_1 = x$ is the solution of homogeneous equation

$$y''(x^2-1) - 2xy' + 2y = 0$$

(b) Find another (non-proportional) solution y_2 of this equation;
(c) Find equation for Wronskian and solve it; calculate Wronskian of y_1; and y_2.
(d) Solve equation

$$y''(x^2-1) - 2xy' + 2y = 1$$

HINT: Try $t = \frac{1}{x^2}$ in difficult integral.

FE-4(*5.5 pts*) (a) Find second order linear homogeneous equation with the coefficient 1 at y'' and with solutions $y_1 = x+1$ and $y_2 = \frac{x+1}{x}$.
(b) Solve this equation with the r.h.e. $\frac{1}{x}$.

FE-5 (*5 pts*) Find the general solutions to (a) and (b):

(a) $$y'' - 3y' + 2y = 2\cosh t$$

(b) $$y'' - 3y' + 2y = \frac{e^{3t}}{e^{2t}+1}$$

FE-6 (*5.5 pts*) Find the general solution

$$\begin{cases} x' = \ \ x + 2y + \dfrac{2}{\cos^3 t} \\ y' = -x - y \end{cases}$$

FE-7 (*3+3 pts*) (a) Find stationary points and classify them for the system

$$\begin{cases} x' = x^1 - 1 \\ y' = y^2 - 1 \end{cases};$$

find separatrices for saddle points.
(b) Sketch the phase picture

FE-8 (*4+4 pts*) (a) Find the periodic trajectory of the system

$$\begin{cases} x' = -x - y + \dfrac{x}{x^2+y^2}\big(2\sqrt{x^2+y^2} + x - y\big) \\ y' = \ \ x - y + \dfrac{y}{x^2+y^2}\big(2\sqrt{x^2+y^2} + x - y\big) \end{cases}$$

and determine the type of it.
(b) Sketch the phase picture. Pay attention to vicinity of singular point $(0,0)$ in which r.h.e. are not defined.
HINT Use polar coordinates

MAT 244S, Final Exam, May 5, 1999. Solutions.
FE-1 Picking $z=y'$ we get linear equation $z'+\frac{2xz}{x^2+1}=x(1+x^2)^{-\frac{3}{2}}$. Solving homogeneous equation

$$z'+\frac{2zx}{x^2+1}=0 \implies \frac{dz}{z}=-\frac{2xdx}{x^2+1} \implies \log z=-\log(x^2+1)+\log C \implies z=\frac{C}{x^2+1}$$

and plugging $z=\frac{C(x)}{x^2+1}$ into non-homogeneous equation we get $C'=\frac{x}{\sqrt{x^2+1}} \implies$

$$C=\sqrt{x^2+1}+\bar{C}_1 \implies z=\frac{1}{\sqrt{x^2+1}}+\frac{\bar{C}_1}{x^2+1} \implies$$
$$y=\int\frac{dx}{\sqrt{x^2+1}}+\bar{C}_1\int\frac{dx}{x^2+1}=\log(x+\sqrt{x^2+1})+\bar{C}_1\arctan x+\bar{C}_2$$

FE-2 Picking $z=y'$ and using $y''=\frac{dz}{dx}=\frac{dy}{dx}\cdot\frac{dz}{dy}=z\frac{dz}{dy}$ we get

$$z\frac{dz}{dy}=2zy \implies dz=2ydy \implies z=y^2+C \implies \frac{dy}{y^2+C}=dx \implies$$
$$x=\begin{cases}\frac{1}{a}\arctan\frac{y}{a}+C_1 & \text{as } C=a^2>0\\ \frac{1}{a}\tanh^{-1}\frac{y}{a}+C_1 & \text{as } C=-a^2<0\\ -\frac{1}{y}+C_1 & \text{as } C=0\end{cases} \implies y=\begin{cases}a\tan(ax+b)\\ a\tanh(ax+b)\\ -\frac{1}{x+b}\end{cases}$$

depending on initial data.

FE-3 (b) Plugging $y_2=zy_1$, $z'=v$ into homogeneous equation we get

$$z''x(x^2-1)+2z'=0 \implies v'x(x^2+1)+2v=0 \implies \frac{dv}{v}=-\frac{2dx}{x(x^2-1)} \implies$$
$$\log v=-\int\frac{2dx}{x(x^2-1)}=-\int\frac{\frac{2dx}{x^3}}{\frac{1}{x^2}-1}=\int\frac{d(\frac{1}{x^2}-1)}{\frac{1}{x^2}-1}=\log\big(-\frac{1}{x^2}+1\big) \implies$$
$$z'=\frac{1}{x^2}-1 \implies z=-\frac{1}{x}-x \implies y_2=(x^2+1)$$

where we picked up some constant arbitrarily.

(c) Equation for Wronskian is $W'=\frac{2xW}{x^2-1}$ which imlpies $\log W=\log(x^2-1)+\log C \implies W=C(x^2-1)$. Calculations show that $W(y_1,y_2)=x^2-1$.

(d) Applying variation of constants we get

$$y=C_1x+C_2(x^2+1)$$

with

$$\begin{cases} C_1'x + C_2'(x^2-1) = 0; \\ C_1' + 2C_2'x = \dfrac{1}{x^2-1} \end{cases} \implies \begin{cases} C_1' = -\dfrac{x^2+1}{(x^2-1)^2}; \\ C_2' = \dfrac{x}{(x^2-1)^2}; \end{cases} \implies$$

$$C_1 = -\int \frac{x^2+1}{(x^2-1)^2}dx = -\frac{1}{2}\int \frac{(x-1)^2+(x+1)^2}{(x^2-1)^2}dx = -\frac{1}{2}\int (\frac{1}{(x-1)^2} + \frac{1}{(x+1)^2})dx =$$

$$\frac{1}{2}(\frac{1}{x-1} + \frac{1}{x+1}) + \bar{C}_1 = \frac{x}{x^2-1} + \bar{C}_1,$$

$$C_2 = \int \frac{xdx}{(x^2-1)^2} = \frac{1}{2}\int \frac{d(x^2-1)}{(x^2-1)^2} = -\frac{1}{2(x^2-1)} + \bar{C}_2$$

$$\implies y = \frac{1}{2} + \bar{C}_1 x + \bar{C}_2(x^2+1)$$

NOTE In this problem errors in (b) doomed (d) and half of (c). Note that since coefficient at y'' is x^2-1 we must divide by it system for C_1', C_2'. Failure to do this caused *0.5 pt* reduction.

Wow! $y = \frac{1}{2}$ is a solution to (d). One coueld guess it.

FE-4 (a) We find the homogeneous linear equation through third order Wronskian $W(y_1, y_2, y) = 0$:

$$\begin{vmatrix} y_1 & y_2 & y \\ y_1' & y_2' & y' \\ y_1'' & y_2'' & y'' \end{vmatrix} = \begin{vmatrix} x+1 & \frac{x+1}{x} & y \\ 1 & -\frac{1}{x^2} & y' \\ 0 & -\frac{2}{x^3} & y'' \end{vmatrix} = -\frac{(x+1)^2}{x^2}y'' + \frac{2(x+1)}{x^3}y' - \frac{2}{x^3}y = 0$$

and since dividing by $-\frac{(x+1)^2}{x^2}$ we get finally

$$y'' - \frac{2}{x(x+1)}y' + \frac{2}{x(x+1)^2}y = 0.$$

(b) To solve

$$y'' - \frac{2}{x(x+1)}y' + \frac{2}{x(x+1)^2}y = \frac{1}{x}$$

we apply the variation of constants

$$y = C_1(x+1) + C_2\frac{(x+1)}{x}$$

leading to

$$\begin{cases} C_1'(x+1) + C_2'\dfrac{(x+1)}{x} = 0, \\ C_1' \qquad + C_2'(-\dfrac{1}{x^2}) = \dfrac{1}{x} \end{cases} \implies C_1' = \frac{1}{x+1}, C_2' = -\frac{x}{x+1} = -1 + \frac{1}{x+1} \implies$$

$$C_1 = \log(x+1) + \bar{C}_1, C_2 = -x + \log(x+1) + \bar{C}_2 \implies$$

$$y = \frac{(x+1)^2}{x}\log(x+1) + (\bar{C}_1 - 1)(x+1) + \bar{C}_2\frac{(x+1)}{x}$$

solves (b).

FE-5 Solving haracteristic equation $\lambda^2 - 3\lambda + 2 = 0$ leads to $\lambda_1 = 1, \lambda_2 = 2$ and to the general solution of homogeneous equation $y^* = C_1 e^t + C_2 e^{2t}$.

To solve (a) we need to find special solutions with r.h.e. $f_1 = e^t$ and $f_2 = e^{-t}$. Since 1 is a simple characteristic root we look at $y_1 = Ate^t$; substituting into equation we get $A = -1, y_1 = -te^t$. Since -1 is not characteristic root $y_2 = Be^{-t}$; substituting into equation we get $B = \frac{1}{6}, y_2 = \frac{1}{6}e^{-t}$. Finally

$$y = -te^t + \frac{1}{6}e^{-t} + C_1 e^t + C_2 e^{2t}$$

solves (a).

To solve (b) we look at $y = C_1 e^t + C_2 e^{2t}$ with variable C_1, C_2. Then

$$\begin{cases} C_1' e^t + \ C' e^{2t} = 0 \\ C_1' e^t + 2C_2' e^{2t} = \dfrac{e^3 t}{e^{2t}+1} \end{cases} \implies C_2' = \frac{e^t}{e^{2t}+1}, C_1' = -\frac{e^{2t}}{e^{2t}+1} \implies$$

$$C_1 = -\int \frac{e^{2t}dt}{e^{2t}+1} = -\frac{1}{2}\int \frac{de^{2t}}{e^{2t}+1} = -\frac{1}{2}\log(e^{2t}+1) + \bar{C}_1,$$

$$C_2 = \int \frac{e^t dt}{e^{2t}+1} = \int \frac{de^t}{e^{2t}+1} = \arctan e^t + \bar{C}_2 \implies$$

$$y = -\frac{1}{2}e^t \log(e^{2t}+1) + e^{2t}\arctan e^t + \bar{C}_1 e^t + \bar{C}_2 e^{2t}.$$

FE-6 Solving the homogeneous system $\mathbf{y}' = \mathcal{A}\mathbf{y}$ with $\mathcal{A} = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$ first. Characteristic equation is $\begin{vmatrix} 1-\lambda & 2 \\ -1 & -1-\lambda \end{vmatrix} = \lambda^2 + 1 = 0$ and $\lambda_{1,2} = \pm i$. Finding eigenvector for $\lambda_1 = i$:

$$\begin{pmatrix} 1-i & 2 \\ -1 & -1-i \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0 \implies \alpha = 2, \beta = -1+i \implies \mathbf{b}_1 = \begin{pmatrix} 2 \\ -1+i \end{pmatrix}$$

(up to the factor) and taking $\mathbf{y} = \begin{pmatrix} 2 \\ -1+i \end{pmatrix}(cost + i\sin t) = \mathbf{y}_1 + i\mathbf{y}_2$ we get $\mathbf{y}_1 = \begin{pmatrix} 2\cos t \\ -\cos t - \sin t \end{pmatrix}$, $\mathbf{y}_2 = \begin{pmatrix} 2\sin t \\ \cos t - \sin t \end{pmatrix}$. So,

$$(\Delta) \qquad \mathbf{y}^* = C_1 \begin{pmatrix} 2\cos t \\ -\cos t - \sin t \end{pmatrix} + C_2 \begin{pmatrix} 2\sin t \\ \cos t - \sin t \end{pmatrix}$$

is the general solution of homogeneous system.

Looking for solution of our system in the form Δ with variable C_1, C_2 we get

$$\begin{aligned}
&C_1'\begin{pmatrix} 2\cos t \\ -\cos t - \sin t \end{pmatrix} + C_2'\begin{pmatrix} 2\sin t \\ \cos t - \sin t \end{pmatrix} = \begin{pmatrix} \frac{2}{\cos^3 t} \\ 0 \end{pmatrix} \implies \\
&C_1' = \frac{\cos t - \sin t}{\cos^3 t} = \frac{1}{\cos^2 t} - \frac{\sin t}{\cos^3 t}, C_2' = \frac{\cos t + \sin t}{\cos^3 t} = \frac{1}{\cos^2 t} + \frac{\sin t}{\cos^3 t} \implies \\
&C_1 = \tan t - \frac{1}{2\cos^2 t} + \bar{C}_1, C_2 = \tan t - \frac{1}{2\cos^2 t} + \bar{C}_2 \implies \\
&\qquad \mathbf{y} = \bigl(\tan t - \frac{1}{2\cos^2 t}\bigr)\begin{pmatrix} 2\cos t \\ -\cos t - \sin t \end{pmatrix} + \bigl(\tan t - \frac{1}{2\cos^2 t}\bigr)\begin{pmatrix} 2\sin t \\ \cos t - \sin t \end{pmatrix} + \mathbf{y}^*
\end{aligned}$$

is the solution.

FE-7 (a) Finding stationary points: $x^2 - 1 = 0, y^2 - 1 = 0$ gives us $(1,1), (1,-1), (-1,1), (-1,-1)$. Finding Jacobi matrix $J = \begin{pmatrix} 2x & 0 \\ 0 & 2y \end{pmatrix}$ and calculating eigenvalues at these points we get $\lambda_1 = \lambda_2 = 1$ (unstable nod) at $(1,1)$, $\lambda_1 = 1, \lambda_2 = -1$ (saddle) at $(1,-1)$, $\lambda_1 = -1, \lambda_2 = 1$ (saddle) at $(-1,1)$, $\lambda_1 = \lambda_2 = -1$ (stable nod) at $(-1,-1)$. Note, that $x = 1$, $x = -1$, $y = 1$ and $y = -1$ are integral curves (actually collections of three integral curves each, separated by stationary points). It gives us all the separatrices in question.

(b) Now phase picture is easy:

Remark Actual equation of integral curves is:

$$\begin{aligned}
\frac{dx}{x^2 - 1} = \frac{dy}{y^2 - 1} \implies \log\frac{x-1}{x+1} + \log C = \log\frac{y-1}{y+1} \implies \frac{y-1}{y+1} = C\frac{x-1}{x+1} \implies \\
y = -\frac{ax - 1}{x - a} = -a + \frac{1 - a^2}{x - a}
\end{aligned}$$

with $a = \frac{C+1}{C-1}$ (the family of hyperbols passing through $(1,1)$ and $(-1,-1)$; as $a = \pm 1$ we get $x = 1$, $x = -1$, $y = 1$ and $y = -1$) and there is one more solution $y = x$ as $C = 1$.

FE-8 Writing the system in the polar coord

$$\rho\rho' = xx' + yy' = -(x^2+y^2) + \bigl(2\sqrt{x^2+y^2} + x - y\bigr) = -\rho^2 + (2+\cos\phi - \sin\phi)\rho,$$
$$\rho^2\phi' = xy' - yx' = (x^2+y^2) = \rho^2 \implies$$
$$(*)\ \begin{cases} \rho' = 2 - \rho + \cos\phi - \sin\phi \\ \phi' = 1 \end{cases} \implies \phi = t + C_0, (\rho - 2 - \cos\phi)' = -(\rho - 2 - \cos\phi)$$

which means that the point rotates uniformly and counter-clockwise around $(0,0)$ and $F = \rho - 2 - \cos\phi = Ce^{-\phi}$ tends to 0. So, there is just one periodic solution: $\rho = 2 + \cos\phi$, $\phi = t + C_0$ and it is stable limit cycle.

(b) Since $\rho + \rho' = 2 + \cos\phi - \sin\phi > 0$ we conclude that ρ increases along trajectories near origin; further, $C < 0$ iff trajectory lies inside of the limit cycle. These trajectories reach origin for a finite number of turns as t decreases and one can call $(0,0)$ an unstable spiral point. On the other hand $C > 0$ iff trajectory lies outside of the limit cycle and it tends to infinity as $t \to -\infty$.