

Chapter 1. Precalculus review

1.8 A Note on mathematical proof; mathematical induction (end)

Induction

The *principle of mathematical induction* is based on the following fundamental property of $\mathbb{N} = \{1, 2, 3, \dots\}$: if we start from 1 and then move from each number to the next one, we cover the whole \mathbb{N} :

If $S \subset \mathbb{N}$ has properties

$$\begin{cases} 1 \in S, \\ \forall k \in S \implies (k+1) \in S \end{cases}$$

then $S = \mathbb{N}$.

or in terms of statements:

If $P(n)$ is the statement about number n

$$\begin{cases} P(1) = \text{true}, \\ \forall k \ P(k) \implies P(k+1) \end{cases}$$

then $\forall k \in \mathbb{N} \ P(k) = \text{true}$.

Consider textbooks examples by yourself. Consider some other examples:

Theorem 1. *If p_k are defined by recurrent equations*

$$(1) \quad p_{k+2} = ap_{k+1} + bp_k \quad k = 1, 2, \dots$$

with initial data

$$(2) \quad p_1 = A, \quad p_2 = B$$

and equation

$$(3) \quad x^2 = ax + b$$

has roots $x_1 \neq x_2$ then

$$(4) \quad p_k = Kx_1^{k-1} + Lx_2^{k-1}$$

with

$$(5) \quad K = \frac{B - Ax_2}{x_1 - x_2}, \quad L = -\frac{B - Ax_1}{x_1 - x_2}$$

We call equations defining p_n in terms of p_m with $m < n$ *recurrent* and extra equations needed to start the process of defining p_n we call *initial data*.

Proof. Equations (5) define K, L so that (2) are defined no matter what $x_1 \neq x_2$ are. Now we need to reformulate the principle of induction:

$$\begin{array}{l} \text{If} \\ \left\{ \begin{array}{l} P(1) = \text{true}, \\ \forall n > 1 \left(\forall k < n \ P(k) \right) \implies P(n) \end{array} \right. \\ \text{then } \forall k \in \mathbb{N} \ P(k) = \text{true} . \end{array}$$

This means that proving $P(n)$ we can use not only that $P(n-1)$ is true but that $P(k)$ is true for all $k = 1, \dots, n-1$. Really, we just apply the previous principle to the composite statement

$$Q(n) = P(1) \& P(2) \& \dots \& P(n) = \left(\forall k < n \ P(k) \right).$$

If p_k with $k \leq n+1$ satisfy (4) then

$$\begin{array}{ll} p_{n+2} = a(Kx_1^n + Lx_2^n) + b(Kx_1^{n-1} + Lx_2^{n-1}) & \text{due to induction and (1)} \\ = Kx_1^{n-1}(ax_1 + b) + Lx_2^{n-1}(ax_2 + b) & \text{simple transformation} \\ = Kx_1^{n-1} \cdot x_1^2 + Lx_2^{n-1} \cdot x_2^2 & \text{due to (3)} \\ = Kx_1^{n+1} + Lx_2^{n+1} & \text{exactly (4) for } k = n+2. \end{array}$$

□

Theorem 2. *If $x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0$ then*

$$(6) \quad \sqrt[n]{x_1 x_2 \dots x_n} \leq \frac{x_1 + x_2 + \dots + x_n}{n}$$

and equality holds iff $x_1 = \dots = x_n$.

The left hand expression is *geometric mean* of x_1, \dots, x_n , the right hand expression is *arithmetic mean* of x_1, \dots, x_n .

Proof. We will skip for simplicity the second statement (about equality).

(i) Note first that (6) holds as $n = 2$. Really $4x_1x_2 = (x_1 + x_2)^2 - (x_1 - x_2)^2 \leq (x_1 + x_2)^2$ and therefore $\sqrt{x_1x_2} \leq \frac{x_1+x_2}{2}$.

(ii) If (6) holds for n then it holds for $2n$. Really,

$$\begin{array}{ll} \sqrt[2n]{x_1 x_2 \dots x_{2n}} \leq \frac{1}{2} \left(\sqrt[n]{x_1 x_2 \dots x_n} + \sqrt[n]{x_{n+1} x_{n+2} \dots x_{2n}} \right) & \text{by (i)} \\ \leq \frac{1}{2} \left(\frac{x_1 + x_2 + \dots + x_n}{n} + \frac{x_{n+1} + x_{n+2} + \dots + x_{2n}}{n} \right) & \text{by conjecture} \\ = \frac{x_1 + x_2 + \dots + x_{2n}}{2n} . & \end{array}$$

Therefore by induction (6) holds for $n = 2^m$, $m = 1, 2, \dots$

(ii) (descent). Let $n < N = 2^m$. Let us take $x_{n+1} = \dots = x_N = a$ where $a = \frac{x_1 + x_2 + \dots + x_n}{n}$ and plug into (6) with n replaced by N . Then we get

$$\sqrt[n]{g^n a^{N-n}} \leq \frac{an + a(n - N)}{N} = a \implies g^n a^{N-n} \leq a^N \implies g^n \leq a^n \implies a \leq g.$$

Theorem 2 is proven in its full generality. □

Nested inductions

Sometimes statement depends on two (or more) numbers and we need to make induction with respect to both:

If $P(n, m)$ is the statement about numbers n and m

$$\begin{cases} P(1, 1) = \text{true}, \\ \forall n, m \left(\forall k < m \ P(n, k) \right) \implies P(n, m) \\ \forall n \left(\forall l < n \ \forall k \leq m \ P(l, k) \right) \implies P(n, 1) \end{cases}$$

then $\forall n \in \mathbb{N} \ \forall m \in \mathbb{N} \ P(n, m) = \text{true}$.

Really: in virtue of first two lines $P(1, m)$ is true for all m ; the third line then proves $P(2, 1)$ thus “capturing a beachhead on level $n = 2$ ”, then the second line proves $P(2, m)$ for all m thus “expanding a beachhead and capturing level $n = 2$ completely” and so on...