Chapter 1. Precalculus review

1.8 A Note on mathematical proof; mathematical induction

Proofs

Mathematicians deal with *statements*, which could be true, false, or neither but all statements should have exact meaning and deal with well-defined objects. Some statements are accepted as *axioms*.
Proof of the *theorem* (whatever it is called: theorem, lemma, proposition, corollary etc, but theorem is the generic name) is a chain of logically correct arguments starting from statements which are true (either axioms or earlier proven theorems) leading to this theorem.
What we consider logical arguments? There arfe few examples (we are not going to study this in the depths):

- If A and $A \implies B$ then B.

  It means that if we proved A and that A implies B than we proved B.

- If $A \implies C$ and $B \implies C$ then $A \vee B \implies C$.

  Forf example if we proved C for all odd integers and also we proved C all even integers, then we proved it for all integers.

- If $A \implies B$ and $\neg B$ then $\neg A$.

  Here $\neg A$ means negation of A. This rule is called *proof by contradiction.* It means that if A implies B and B is false, then A is also false.

  Note that if $A \implies B$ and B is true then we cannot say anything about A. On the other hand in many sciences if conjecture A implies many completely different corollaries which are true (confirmed by an experiment) then A is considered to be a good conjecture and could be promoted to the rank of theory. From the logical point of view this is a fallacy: experiments do not prove the conjecture but simply kill competing conjectures - but this is a way of physics etc and in the certain degree of mathematics too (suggesting that some statement is true and thus encouraging to prove it). Very interesting books:

  Mathematics and Plausible Reasoning (Volumes I, II) by George Polya.

- Let theorem state that $A \implies B$. Then $B \implies A$ is called *converse* theorem. Theorem $A \iff B$ means exactly that $A \implies B$ and $B \implies A$ and often its proof consists of two parts (direct and converse). Surely it could be that

  - Both direct and converse statements are true; then A is *the necessary and sufficient condition* for B [or statements A and B are *equivalent*],

  - Only $A \implies B$ is true; then A is *sufficient (but not necessary) condition* for B [or statement A is *stronger* than B],
  - Only $B \implies A$ is true; then A is *necessary (but not sufficient) condition* for B [or statement A is *weaker* than B],
  - Neither $A \implies B$ nor $B \implies A$ are true; then A is *neither necessary nor sufficient condition* for B.

- Often statement is $\forall x \in X\ A(x)$ which reads as "every element of X has property $A(x)$". Negation of this is

  $$\neg\Bigl(\forall x \in X\ A(x)\Bigr) = \exists x \in X\ \neg A(x)$$

  which reads: "exists an element x of X which does not have property $A(x)$.

  Such x is called *counterexample*. F.e. 9 is a counterexample to the statement: every odd number is a prime. On the other hand the negation of the statement $\exists x \in X\ A(x)$ is $\forall x \in X\ \neg A(x)$. If we want to prove that there exists no rational number r such that $r^2 = 2$ we need to prove that for every rational number $r^2 \neq 2$. To counter the assertion "all sheeps are white" one should bring a black one but even a huge herd of black sheeps does not counter assertion that "there exists a white sheep".

- Sometimes in logical statements two $\forall$ are together and we can permute them; or we can permute two $\exists$. However permutting of $\forall$ and $\exists$ is not possible: statement $\exists x\ \forall y\ A(x,y)$ is stronger than $\forall y\ \exists x\ A(x,y)$, Really: the statement "for each lock there is a key which opens it" is weaker than "exists a key which opens every lock". The second statement asserts the existence of the master-key.