Chapter 10. Improper integrals

Notes

The notion of integral is one of the most important in mathematics and there are many definitions of integral to cover more and more general classes of functions and even more general objects. In the second year calculus you will study multidimensional integrals.

There are two ways to generalize:

- Completely rewamping the current definition;
- Expanding the current definition.

The improper integrals are introduced in the second way. We want to define

$$\int_{a^*}^{b^*} f(x)\,dx$$

while this integral is not defined in the "normal" way because of one or more following reasons:

(i) $a^* = -\infty$;

(ii) $b^* = +\infty$;

(iii) $a^* = a > -\infty$ but $f(x)$ is unbounded as $x \to a^+$;

(iv) $b^* = b < \infty$ but $f(x)$ is unbounded as $x \to b^-$;

(v) $f(x)$ is unbounded as $x \to c_j$, $j = 1, \dots, N$ where $a^* < c_1 < c_2 < \cdots < c_N < b^*$.

In what follows a, b, ... are usual numbers and a^*, b^* could be also $-\infty$, $+\infty$.

The general idea

(1) Define improper integral by taking a limit in cases (i)-(iv), when there is exactly one "bad" point.

(2) Then cover the general case just by splitting integral.

Cases (i),(ii)

Definition 1.

(i) $\displaystyle\int_{-\infty}^{b} f(x)\,dx \overset{\text{def}}{=} \lim_{N\to-\infty}\int_{N}^{b} f(x)\,dx$ provided

- $\int_N^b f(x)\,dx$ exists for all $N<b$;
- The limit also exists.

(ii) $\displaystyle\int_{a}^{\infty} f(x)\,dx \overset{\text{def}}{=} \lim_{N\to+\infty}\int_{a}^{N} f(x)\,dx$ provided

- $\int_a^N f(x)\,dx$ exists for all $N>a$;
- The limit also exists.

One can prove that such integrals have all the usual properties including

$$\int_{a^*}^{b^*} f(x)\,dx = \int_{a^*}^{c} f(x)\,dx + \int_{c}^{b^*} f(x)\,dx \tag{1}$$

as $a^* < c < b^*$ under temporary assumption that only one of a^*, b^* is infinite.

Cases (iii),(iv)

Definition 2.

(i) $\displaystyle\int_{a}^{b} f(x)\,dx \overset{\text{def}}{=} \lim_{\epsilon\to0^+}\int_{a+\epsilon}^{b} f(x)\,dx$ provided

- $\int_{a+\epsilon}^b f(x)\,dx$ exists for all $\epsilon\in(0,b-a)$;
- The limit also exists.

(ii) $\displaystyle\int_{a}^{b} f(x)\,dx \overset{\text{def}}{=} \lim_{\epsilon\to0^+}\int_{a}^{b-\epsilon} f(x)\,dx$ provided

- $\int_a^{b-\epsilon} f(x)\,dx$ exists for all $\epsilon\in(0,b-a)$;
- The limit also exists.

One can prove that such integrals have all the usual properties including (1) as $a<c<b$ under temporary assumption that only one of a, b is a “bad” point.

Synthesis. I

Definition 3. Let both endpoints a^*, b^* be bad "bad" (so either $a^* = -\infty$ or $f(x)$ is unbounded as $x \to a^+$ and also either $b^* = +\infty$ or $f(x)$ is unbounded as $x \to b^-$; then as long as all inner points are good in the sense that $\int_\alpha^\beta f(x)\,dx$ exists in the "normal" sense for all $\alpha, \beta \in (a, b)$, then

$$\int_{a^*}^{b^*} f(x)\,dx \stackrel{\text{def}}{=} \int_{a^*}^{\gamma} f(x)\,dx + \int_{\gamma}^{b^*} f(x)\,dx$$

where $\gamma \in (a, b)$ is arbitrary.

Remark 4. (i) It follows from (1) that in the definition 3 $\displaystyle\int_{a^*}^{b^*} f(x)\,dx$ is well-defined (does not depend on the choice of γ;

(ii) One could define

$$\int_{-\infty}^{+\infty} f(x)\,dx \stackrel{\text{def}}{=} \lim_{\substack{\alpha\to-\infty\\ \beta\to+\infty}} \int_{\alpha}^{\beta} f(x)\,dx,$$

$$\int_{a}^{b} f(x)\,dx \stackrel{\text{def}}{=} \lim_{\substack{\alpha\to a^+\\ \beta\to b^-}} \int_{\alpha}^{\beta} f(x)\,dx,$$

$$\int_{a}^{+\infty} f(x)\,dx \stackrel{\text{def}}{=} \lim_{\substack{\alpha\to a^+\\ \beta\to+\infty}} \int_{\alpha}^{\beta} f(x)\,dx,$$

$$\int_{-\infty}^{b} f(x)\,dx \stackrel{\text{def}}{=} \lim_{\substack{\alpha\to-\infty\\ \beta\to b^-}} \int_{\alpha}^{\beta} f(x)\,dx$$

but currently we do not know what double limit is.

Synthesis. I

Definition 5. In the general case

$$\int_{a^*}^{b^*} f(x)\,dx \stackrel{\text{def}}{=} \int_{a^*}^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \ldots \int_{c_{N-1}}^{c_N} f(x)\,dx + \int_{c_N}^{b^*} f(x)\,dx.$$

Final remarks

Remark 6. One could expand definitions 1, 2 in the case of the infinite limit; then one can expand definitions 3, 5 the same way assuming that we add only infinities of the same sign and also the finite numbers.

Definition 7. Integral is absolutely converging and f is absolutely integrable if
$$\int_{a^*}^{b^*} |f(x)|\,dx < \infty.$$

Remark 8. (i) Assume that $\int_a^b f(x)\,dx$ does not exists because there is exactly one point $c \in (a,b)$; further assume that $\int_a^b f(x)\,dx$ is not defined even as an improper integral because either $\int_a^c f(x)\,dx$ is not defined, or $\int_c^b f(x)\,dx$ is not defined, or one of them is $+\infty$ and another is $-\infty$. Then one can define
$$\text{ess} \int_a^b f(x)\,dx \stackrel{\text{def}}{=} \lim_{\epsilon\to+0}\Bigl(\int_a^{c-\epsilon} f(x)\,dx + \int_{c+\epsilon}^b f(x)\,dx\Bigr);$$
this definition is useful but it is substantially weaker than the definition we use.

(ii) Assume that there are no "bad" points on $(-\infty,+\infty)$ but $\int_{-\infty}^{\infty} f(x)\,dx$ is not defined as an improper integral because either $\int_{-\infty}^0 f(x)\,dx$ is not defined, or $\int_0^\infty f(x)\,dx$ is not defined, or one of them is $+\infty$ and another is $-\infty$. Then one can define
$$\text{ess} \int_{-\infty}^{\infty} f(x)\,dx \stackrel{\text{def}}{=} \lim_{N\to+\infty} \int_{-N}^{N} f(x)\,dx;$$
this definition is useful but it is substantially weaker than the definition we use.