Chapter 7

7.6 Inverse Trigonometric Functions

Defintion

Obviously equations $\sin x = t$, $\cos x = t$, $\tan x = t$ have either no solution or infinitely many of them we need defining inverse function restrict the domain of the original function. For $\sin x$ and $\csc x$ such natural domains are $[-\pi/2, \pi/2]$ and $(-\pi/2, \pi/2)$ respectively; for $\cos x$ and $\sec x$ such domains are $[0, \pi]$ and $(0, \pi)$; finally for $\tan x$ and $\cot x$ we can pick up domains $(-\pi/2, \pi 2)$ and $(0, \pi)$ respectively:

Definition 1. We introduce uniquely the following trigonometric functions:

- $\arcsin : [-1, 1] \to (-\pi/2, \pi/2) \ni y = \arcsin(x)$ iff $\sin x = y$;
- $\arccos : [-1, 1] \to (0, \pi) \ni y = \arccos(x)$ iff $\cos x = y$;
- $\text{arccsc} : (-\infty, -1] \cup [1, +\infty] \to (-\pi/2, \pi/2) \ni y = \text{arccsc}(x)$ iff $\csc x = y$;
- $\text{arcsec} : (-\infty, -1] \cup [1, +\infty] \to (0, \pi) \ni y = \text{arcsec}(x)$ iff $\sec x = y$;
- $\arctan : (-\infty, +\infty) \to (-\pi/2, \pi/2) \ni y = \arctan(x)$ iff $\tan x = y$;
- $\text{arccot} : (-\infty, +\infty) \to (0, \pi) \ni y = \arctan(x)$ iff $\cot x = y$;

Remark 2. Texbook notation $\sin^{-1} x$ is consistent with set theory but not consistent with notations like $\sin^2 x$ and thus leads to a possible confusion.

Differentiation

Theorem 3. *Inverse trigonometric function have derivatives*

$$\big(\arcsin(x)\big)' = -\big(\arccos(x)\big)' = \frac{1}{\sqrt{1-x^2}}, \tag{1}$$

$$-\big(\text{arcsec}(x)\big)' = \big(\text{arccsc}(x)\big)' = \frac{1}{|x|\sqrt{x^2-1}}, \tag{2}$$

$$\big(\arctan(x)\big)' = -\big(\text{arccot}(x)\big)' = \frac{1}{1+x^2}. \tag{3}$$

Proof. As $y = \arcsin x$, $x = \sin y$ we have

$$\frac{d\arcsin(x)}{dx} = \frac{dy}{dx} = (\frac{dx}{dy})^{-1} = (\cos y)^{-1} = \frac{1}{\sqrt{1-x^2}}$$

since in the interval in question $\cos x > 0$.

Since $\arccos(x) = \frac{\pi}{2} - \arcsin(x)$ we get $\dfrac{d\arcsin x}{dx} = -\dfrac{1}{\sqrt{1-x^2}}$.

Since $\mathrm{arccsc} x = \arcsin \dfrac{1}{x}$

$$(\mathrm{arccsc}(x))' = \frac{1}{\sqrt{1-\frac{1}{x^2}}} \times \Big(-\frac{1}{x^2}\Big) = -\frac{1}{|x|\sqrt{x^2-1}}.$$

Since $\mathrm{arcsec}(x) = \frac{\pi}{2} - \mathrm{arccsc}(x)$ we get $-(\mathrm{arccsc} x)' = \dfrac{1}{|x|\sqrt{x^2-1}}$.

Finally, if $y = \arctan x$, $x = \tan y$ we get $\cos^{-2} y = 1 + x^2 \implies \cos^2 y = \dfrac{1}{1+y^2} \implies \cos(2y) = \dfrac{1-x^2}{1+x^2}$ and therefore $\arctan(x) = \dfrac{1}{2}\arccos\dfrac{1-x^2}{1+x^2}$. Then

$$(\arctan(x))' = \frac{1}{2} \cdot \frac{1}{\sqrt{1-\Big(\frac{1-x^2}{1+x^2}\Big)^2}} \times \Big(\frac{1-x^2}{1+x^2}\Big)' = \frac{1}{1+x^2}.$$

Since $\mathrm{arccot}(x) = \frac{\pi}{2} - \mathrm{arccot}(x)$ we get the last equality. □

Remark 4. (i) Other proof of derivative $\arctan(x)$:

$$\frac{dy}{dx} = (\frac{dx}{dy})^{-1} = \cos^2 y = \frac{1}{1+x^2}.$$

(ii) We defined trigonometric functions in a bit fuzzy way because we referred to measuring of arcs. One could introduce $\arctan x$ as

$$\arctan(x) = \int_0^x \frac{1}{1+t^2}\,dt,$$

then introduce $\tan x$ on $(-\pi/2, \pi/2)$ as an inverse function, then extend it by periodicity and then introduce all other trigonometric and inverse trigonometric functions.

Graphs of Trigonometric Functions

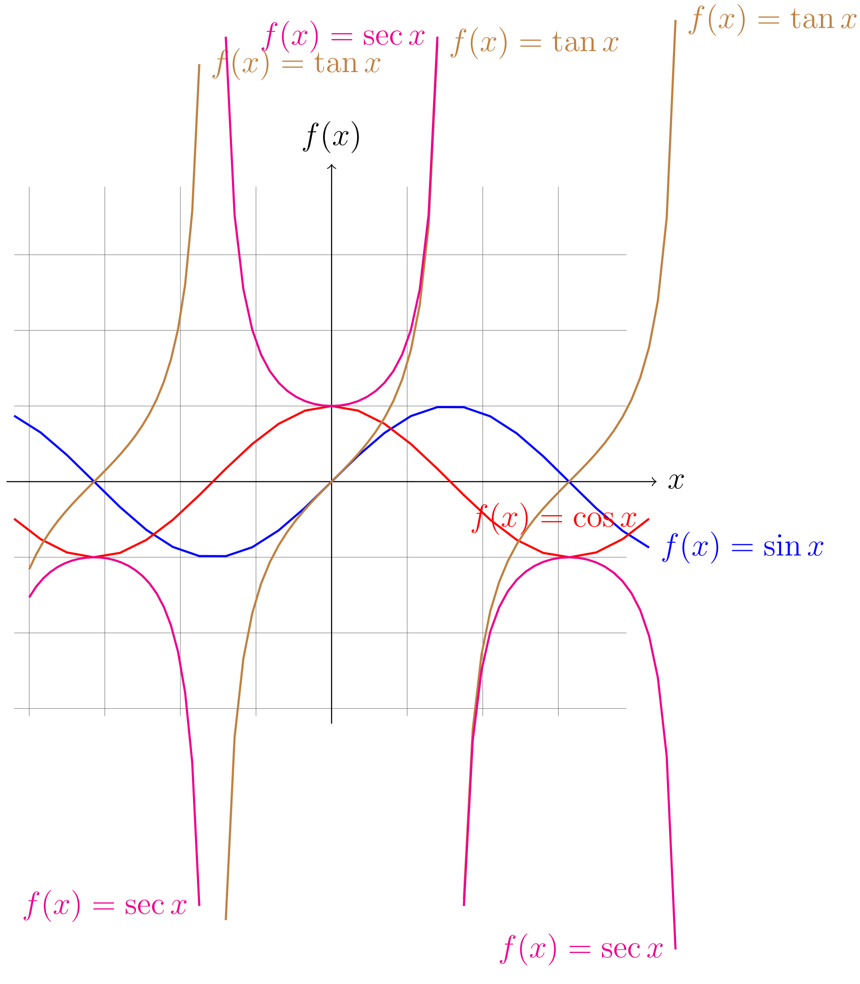

Figure 1: Trigonometric functions