Chapter 7

7.4 The Exponent Function. I

Definition

We start from axiomatic definition of exponent:

Definition 1. We call exponent function $\exp(x)$ defined on $(-\infty, +\infty)$ which is invese to logarithm:

$$\log \exp(x) = x \quad \forall x, \qquad \exp(\log x) = x \quad \forall x > 0. \tag{1}$$

Note that for rational x $\exp(r) = e^r$:

Theorem 2. *We call exponent function* $\exp(x)$ *defined on* $(-\infty, +\infty)$ *which is invese to logarithm:*

$$\exp(r) = e^r. \tag{2}$$

Proof. Since $\log e^r = r \log e = r$ we conclude that $e^r = \exp(r)$. □

So, for rational r exp is a power of e. From now on we use e^x as another notation for $\exp(x)$.

The following properties follow immediately from the properties of logarithm:

$$e^0 = 1, \qquad e^1 = e, \tag{3}$$

$$e^{x+y} = e^x \cdot e^y, \tag{4}$$

$$e^{rx} = (e^x)^r \tag{5}$$

for $r \in \mathbb{Q}$.

Theorem 3.

$$(e^x)' = e^x. \tag{6}$$

Proof. By inverse function rule as $y = e^x$, $x = \log y$ and

$$\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1} = \Big((\log y)'\Big)^{-1} = \left(\frac{1}{y}\right)^{-1} = y = e^x.$$

□

Also exponent is very fast growing as $x \to +\infty$ and fast decaying as $x \to -\infty$:

Theorem 4. *(i) Exponent is very fast growing as $x \to +\infty$ $x \to +\infty$:*

$$\lim_{x \to +\infty} \frac{e^x}{x^N} = 0 \tag{7}$$

for arbitrarily large N;
(ii) Exponent is very fast growing as $x \to +\infty$:

$$\lim_{x \to -\infty} x^N e^x = 0 \tag{8}$$

for arbitrarily large $\alpha > 0$.

Proof. By L'Hopital rule. Another proof: from Theorem 8 of the previous lecture. □

There are few functions related to e^x:

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \text{cosine hyperbolic,} \tag{9}$$

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \text{sine hyperbolic,} \tag{10}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \text{tan hyperbolic.} \tag{11}$$

One can prove their properties:

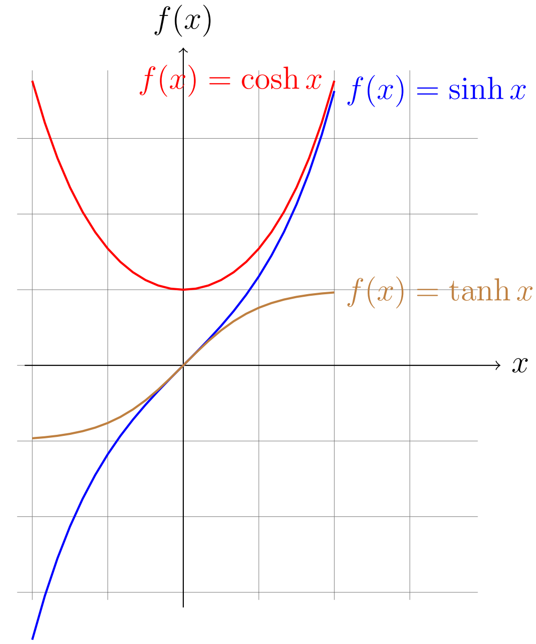
$$(\cosh x)^2 - (\sinh x)^2 = 1, \tag{12}$$

$$(\cosh x)' = \sinh x, \tag{13}$$

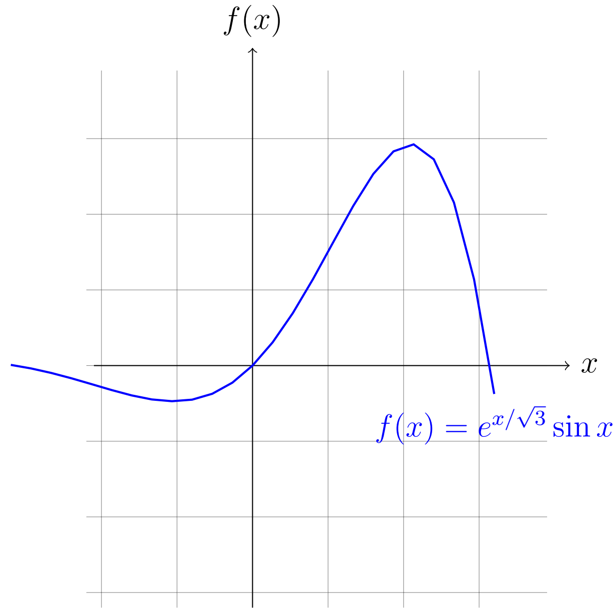
$$(\sinh x)' = \cosh x, \tag{14}$$

$$(\tanh x)' = (\cosh x)^{-2}. \tag{15}$$

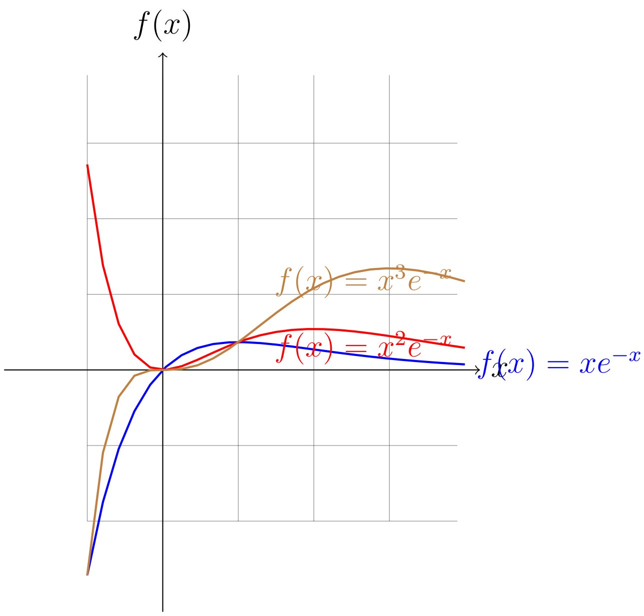
Some Graphs



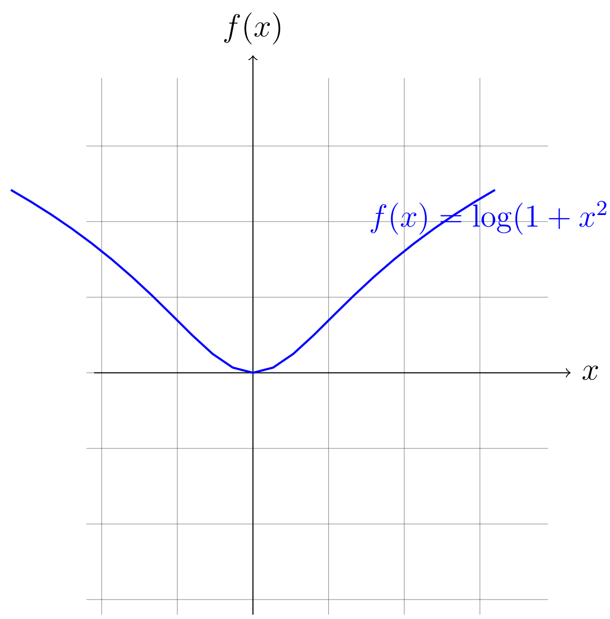
Hyperbolic functions



$f(x) = e^{x/\sqrt{3}} \sin x$



$f(x) = x^n e^{-x}$



$f(x) = \log(1 + x^2)$

7.5 The Exponent Function. II

Definition of a^x

As $a > 0$ we define a^x:

Definition 5.

$$E_a(x) = e^{x \log a}. \tag{16}$$

Then it follows from this definition and definition of $\log_a y$ that

$$E_a(x) = y \iff x = \log_a y. \tag{17}$$

Further, as r rational it follows from the properties of e^x that $E_a(r) = e^{r \log a} = (e^{\log a})^r = a^r$ where a^r is a power. From now on notation $E_a(x)$ is replaced by a^x.

Then it follows from the property of exponent that

$$a^0 = 1, \qquad a^1 = a, \tag{18}$$

$$a^{x+y} = a^x \cdot a^y, \tag{19}$$

$$a^{xy} = (a^x)^y, \tag{20}$$

$$(ab)^x = a^x \cdot b^x. \tag{21}$$

Derivative of u^v

Note that

$$(a^x)' = (e^{x \log a})' = e^{x \log a}) \cdot (x \log a)' = a^x \log a.$$

for $a = \text{const} > 0$ and

$$(u^v)' = (e^{v \log u})' = e^{v \log u}) \cdot (v \log u)' = u^v \cdot \bigl(v' \log u + v \cdot (\log u)'\bigr) = \\ u^v \cdot \bigl(v' \log u + v \cdot \frac{u'}{u}\bigr) = u^v v' \log u + u^{v-1} u'.$$

So

$$(a^x)' = a^x \log a. \tag{22}$$

$$(u^v)' = u^v v' \log u + u^{v-1} u'. \tag{23}$$