Chapter 7

7.2 The Logarithm Function. I

7.2.1 Definition

We follow our textbook with little simplifications. This approach is opposite to one of most textbooks.

Check the updated table of integrals from the textbook (coming in handouts soon)

Definition

We start from axiomatic definition of logarithm:

Definition 1. We call logarithm any function $L(x)$ defined on $(0, +\infty)$ which satisfies

$$L(xy) = L(x) + L(y) \qquad \forall x > 0,\ y > 0 \tag{1}$$

and is not identically 0.

Such approach raises questions: Do such functions exist? How many such function exist? What are their properties?

We start from the last one. Plugging into (1) $x = y = 1$ we conclude that

$$L(1) = L(1) + L(1) \implies L(1) = 0.$$

Then plugging in (1) $y = 1/x$ we conclude that

$$0 = L(1) = L(x) + L(1/x) \implies L(1/x) = -L(x).$$

Then plugging $y := 1/y$ we conclude that

$$L(x/y) = L(x) + L(1/y) = L(x) - L(y).$$

Derivative

Now we want to calculate derivative which is limit of

$$\frac{L(x+h)-L(x)}{h} = \frac{1}{h}L\big(\frac{x+h}{x}\big) = \frac{1}{x}\cdot\frac{x}{h}L\big(1+\frac{h}{x}\big) = \frac{1}{x}\cdot\frac{1}{\hbar}\Big(L\big(1+\hbar\big)-L\big(1\big)\Big)$$

with $\hbar = h/x$. Then $\hbar \to 0$ as $h \to 0$ and the right-hand expression tends to $\frac{1}{x}L'(1)$. Therefore

$$L'(x) = \frac{A}{x}. \tag{2}$$

Then

$$L(x) = A\int \frac{dx}{x} = \int_1^x \frac{dt}{t} + C.$$

Since $L(1) = 0$ we conclude that $C = 0$:

$$L(x) = A\int_1^x \frac{dt}{t}.$$

Logarithm redefined. Some properties

We can redefine logarithm based on this with $A = 1$ (all other such functions would differ by a constant factor):

Definition 2. We call (a natural) logarithm function $l(x)$ defined on $(0,+\infty)$ by

$$l(x) = \int_1^x \frac{dt}{t}. \tag{3}$$

Remark 3. We know that such integral (3) exists as $x > 0$ since on $(0,\infty)$ $1/x$ is continuous function. We need to prove (1).

Theorem 4. *Function $l(x)$ defined by (3) satisfies (1).*

Proof. Consider

$$l(xy) = \int_1^{xy} \frac{dt}{t} = \int_1^x \frac{dt}{t} + \int_x^{xy} \frac{dt}{t}.$$

Changing in the second term variables $t = xz$ with z ranging from 1 to y we get

$$l(xy) = \int_1^{xy} \frac{dt}{t} = \int_1^x \frac{dt}{t} + \int_1^y \frac{du}{u} = l(x) + l(y).$$

$\square$

Theorem 5. *Function $l(x)$ defined by (3) satisfies*

$$l\bigl(x^{p/q}\bigr) = \frac{p}{q} l(x) \tag{4}$$

forall $x > 0$ and $p/q \in \mathbb{Q}$.

Proof. Note that

$$l(x^p) = l(\underbrace{x \cdot x \cdots x}_{p \text{ factors}}) = \underbrace{l(x) + l(x) + \cdots + l(x)}_{p \text{ terms}} = pl(x)$$

and (4) is proven as $q = 1$, $p > 0$.

Then as $p < 0$ $l(x^p) = -l(1/x^p) = -l(x^{-p}) = -(-p)l(x) = pl(x)$; as $p = 0$ $l(x^0) = l(1) = 0$ and $0 \cdot l(0) = 0$ as well. So (4) is proven for all $p \in \mathbb{Z}$, $q = 1$.

Plugging $x = y^{1/q}$, $p = q$ and $x^q = y$ we conclude that $l(y) = ql(y^{1/q})$ and therefore $l(y^{1/q} = \frac{1}{q} l(y)$; so (4) is proven for all $q \in \mathbb{Z}$, $q \neq 0$, $p = 1$.

Finally, $l(x^{p/q}) = l\bigl((x^{1/q})^p\bigr) = pl(x^{1/q}) = p \cdot \frac{1}{q} l(x)$. □

Theorem 6. *Function $l(x)$ defined by (3) is monotone increasing function on $(0, +\infty)$ with range $(-\infty, +\infty)$: as $l(x) \to -\infty$ as $x \to 0^+$ and $l(x) \to +\infty$ as $x \to +\infty$.*

Proof. $l(x)$ is differentiable due to property of integral and it is monotone increasing since $l'(x) > 0$.

So, $l(2) > l(1) = 0$. Further, $l(x) \geq l(2^n)$ as $x \geq 2^n$. Defining $n = n(x)$ as the maximal number $n \in \mathbb{N}$ such that $2^n \leq x$ we conclude that $l(x) \geq n(x) \log 2$. The latter expression tends to $+\infty$ as $x \to +\infty$. So $l(x) \to +\infty$ as $x \to +\infty$.

Now $l(x) = -l(1/x)$ and

$$x \to 0^+ \implies 1/x \to +\infty \implies l(1/x) \to +\infty \implies l(x) = -l(1/x) \to -\infty.$$

□

Number e

From theorem 5 follows that for each $y \in (-\infty, +\infty)$ exist exactly one $x > 0$ such that $l(x) = y$. In particular this happens as $y = 1$:

Definition 7. Number e is the (only) number such that $l(e) = 1$.

It follows immediately that $e > 1$.

From now on we denote $l(x)$ as $\log x$ aka $log_e x$ aka $\ln x$. Let me repeat its properties:

$$\ln 1 = 0, \qquad \ln e = 1, \tag{5}$$

$$ln(xy) = \ln x + \ln y, \tag{6}$$

$$\ln(1/x) = -\ln x, \tag{7}$$

$$\ln(x/y) = \ln x - \ln y, \tag{8}$$

$$\ln x^r = r \ln x \tag{9}$$

as $x > 0, y > 0$, $r \in \mathbb{Q}$.

I remind that $(\ln x)' = 1/x$ and therefore we can fill $\int x^\alpha \, dx = \ln x + C$ as $\alpha = -1$ instead of $(\alpha+1)^{-1} x^{\alpha+1}$ as $\alpha \ne -1$.

7.3 The Logarithm Function. II

There are no much theory except:

$$(\log |x|)' = \frac{1}{|x|}(|x|)' = \frac{1}{x} \qquad \text{as } x \ne 0 \tag{10}$$

and thus

$$\int \frac{dx}{x} = \ln |x| + C \qquad \text{as } x \ne 0. \tag{11}$$

However two branches of antiderivative (as $x < 0$ and $x > 0$) are not connected.

Also one should note that

$$\int \frac{u'}{u} \, dx = \ln |u| + C \qquad \text{as } u \ne 0. \tag{12}$$

Also logarithm is very slowly increasing as $x \to +\infty$ or as $x \to 0^+$:

Theorem 8. *(i) Logarithm is very slowly increasing as $x \to +\infty$:*

$$\lim_{x \to +\infty} \frac{\log x}{x^\alpha} = 0 \tag{13}$$

for arbitrarily small $\alpha > 0$;
(ii) Logarithm is very slowly increasing as $x \to 0^+$:

$$\lim_{x \to 0^+} \frac{\log x}{x^{-\alpha}} = 0 \tag{14}$$

for arbitrarily small $\alpha > 0$.

Proof. By L'Hopital rule

$$\lim_{x\to+\infty}\frac{\log x}{x^\alpha}=\lim_{x\to+\infty}\frac{\dfrac{1}{x}}{\alpha x^{\alpha-1}}=\lim_{x\to+\infty}\frac{\dfrac{1}{x}}{\alpha x^{\alpha-1}}=\lim_{x\to+\infty}\frac{1}{\alpha}x^{-\alpha}=0$$

and then plugging $x = 1/y$

$$\lim_{y\to 0^+}\frac{-\log y}{y^{+\alpha}}=0.$$

□

Definition 9. As $x > 0$, $y > 0, y \neq 1$

$$\log_y x = \frac{\log x}{\log y}. \tag{15}$$