Chapter 5

5.3 – 5.4 Properties of integral

Due to the theorem 7 of the previous lecture we can define integral as

How integral depends on function

Definition 1. $S = \int_a^b f(x)\,dx$ iff

$$\forall \epsilon > 0 \exists \delta > 0 \max_j \Delta_j \le \delta \implies |\mathrm{RS}(I, f; \mathcal{P}, \xi_1, \dots, \xi_n) - S| \le \epsilon. \tag{1}$$

Now immediately

Theorem 2.

$$\int_a^b \big(f(x) \pm g(x)\big)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx, \tag{2}$$

$$\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx \tag{3}$$

and the left-hand expressions exist provided the right-hand expressions exist.

Proof. If we consider Riemannian sums with the same partitions and choice of points $\xi_1, \dots, \xi_n$ then (2),(3) hold for them. Then taking limits as $\max_j \Delta_j \to 0$ leads to (2), (3) for integrals. □

Theorem 3. *Let $f(x) \ge g(x)$ for all $x \in [a, b]$. Then*

$$\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx \tag{4}$$

provided both integral exist.

Proof. If we consider Riemannian sums with the same partitions and choice of points $\xi_1, \dots, \xi_n$ then (4) holds for them. Then taking limits as $\max_j \Delta_j \to 0$ leads to (4) for integrals. □

Theorem 3 implies that

$$|\int_a^b f(x)\,dx| \le \int_a^b |f(x)|\,dx \tag{5}$$

since $-|f(x)| \le f(x) \le |f(x)|$.

Note that

$$\int_a^b dx = (b-a) \tag{6}$$

since obviously $\mathrm{RS}(a,b,1;\mathcal{P},\xi_1,\dots,\xi_n) = \sum_{j=1}^n \Delta_j = (b-a)$.

Theorem 4. *(Mean value theorem) (i)*

$$\min_{a\le x\le b} f(x) \le \frac{1}{(b-a)}\int_a^b f(x)\,dx \le \max_{a\le x\le b} f(x). \tag{7}$$

(ii) Further, if $w(x)\ge 0$ on $[a,b]$ then

$$\min_{a\le x\le b} f(x) \le \frac{1}{\int_a^b w(x)\,dx}\int_a^b f(x)w(x)\,dx \le \max_{a\le x\le b} f(x) \tag{8}$$

provided $\int_a^b w(x)\,dx \ne 0$.

Proof. Let us prove (ii); (i) is its special case as $w(x)=1$ (for $w=1$ denominator is $(b-a)$. Then since

$$\min_{a\le\xi\le b} f(\xi)\cdot w(x) \le f(x)w(x) \le \max_{a\le\xi\le b} f(\xi)\cdot w(x)$$

theorem 3 implies that

$$\min_{a\le\xi\le b} f(\xi)\int_a^b w(x)\,dx \le \int_a^b f(x)w(x)\,dx \le \max_{a\le\xi\le b} f(\xi)\int_a^b w(x)\,dx$$

and since $\int_a^b w(x)\,dx \ge 0$ we arrive to (8). □

Definition 5. The expression in the middle of (7) is a *mean value of $f(x)$ on $[a,b]$*. The expression in the middle of (8) is a *weighted mean value of $f(x)$ on $[a,b]$ with the weight $w(x)$*.

How integral depends on interval

Theorem 6. *If $f(x)$ integrable on $[a,b]$ then it is integrable on $[\alpha,\beta]$ as $a\le \alpha<\beta\le b$.*

Proof. Consider some partition $\mathcal{P}$ on $[\alpha, \beta]$ with $\max_j \Delta_j \le \delta$. We can extend it to partition $\mathcal{P}'$ on $[a, b]$ satisfying the same inequalities. Then according to theorem 8 of the previous lecture

$$\sum_{j=1}^{n} \operatorname{osc}_{I_j} f \cdot \Delta_j \le \epsilon \tag{9}$$

where

$$\operatorname{osc}_J f = \max_J f - \min_J f \tag{10}$$

and I_j runs through partition $\mathcal{P}'$; removing some (non-negative) terms we conclude that (9) holds for partition $\mathcal{P}$ as well.

Applying theorem 8 of the previous lecture we conclude that f is integrable on $[\alpha, \beta]$. ☐

Theorem 7.

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx \tag{11}$$

as $a < b < c$ and the lef-hand expression exists iff the rightf-hand expressions exist.

Proof. If the left-hand expression exists then both right-hand expression exist as well due to theorem 6. Conversely, consider some partition $\mathcal{P}$ on $[a, c]$. It defines partitions $\mathcal{P}'$ and $\mathcal{P}''$ on $[a, b]$ and $[b, c]$. Little problem: $\mathcal{P}$ does not necessarily contain point b so $\mathcal{P}' \cup \mathcal{P}''$ could contain an extra point in comparison with $\mathcal{P}$.

However, even then the difference between $\mathrm{RS}(a, c, f; \mathcal{P}, \xi_1, \ldots, \xi_n)$ and

$$\mathrm{RS}(a, b, f; \mathcal{P}', \xi_1, \ldots, \xi'_m) + \mathrm{RS}(b, c, f; \mathcal{P}'', \xi''_m, \ldots, \xi_n)$$

does not exceed $2M\delta$ (where ξ'_m and ξ''_m are two points on the interval I_j covering b). Then one can take $\delta > 0$ such that

$$\begin{aligned} &|\mathrm{RS}(a, b, f; \mathcal{P}', \xi_1, \ldots, \xi'_m) - \int_a^b f(x)\,dx| \le \epsilon/3, \\ &|\mathrm{RS}(b, c, f; \mathcal{P}'', \xi''_m, \ldots, \xi_n) - \int_b^c f(x)\,dx| \le \epsilon/3 \end{aligned}$$

and $2M\delta \le\le \epsilon/3$; then

$$|\mathrm{RS}(a, c, f; \mathcal{P}, \xi_1, \ldots, \xi_n) - \Bigl(\int_a^b f(x)\,dx + \int_b^c f(x)\,dx\Bigr)| \le \epsilon.$$

Since it holds for any ϵ provided $\max_j \Delta_j \le \delta = \delta(\epsilon)$ we conclude (11) holds. ☐

Definition 8. Function f is piecewise-continuous on $[a, b]$ if it continuous except finite number points $a < \alpha_1 < \cdots < \alpha_N < b$ and at every poin α_j one-sided limits $\lim_{x\to\alpha_j^\pm} f(x)$ exist.

Corollary 9. *Function f piecewise-continuous on $[a, b]$ is integrable on $[a, b]$.*

Proof. Due to theorem 7 and integrability of continuous functions. □

Let us extend definition of integral:

Definition 10.

$$\int_a^b f(x)\,dx = \begin{cases} 0 & \text{as } \ b = a, \\ -\displaystyle\int_b^a f(x)\,dx & \text{as } \ b < a. \end{cases} \tag{12}$$

Theorem 6 implies that

Theorem 11. *Equalities (2), (3), (11) remain true for all a, b, c (without assumption $a < b < c$).*

How integral depends on its upper limit

Let f be integrable on $[a, b]$. Consider function $F(t)$ defined on $[a, b]$ by

$$F(t) = \int_a^t f(x)\,dx. \tag{13}$$

Theorem 12. *(i) Function $F(t)$ is continuous function of t.*

(ii) Further, if f is continuous at point $x = c$ then F is differentiable at point c and $F'(c) = f(c)$.

Proof. If f is differentiable on $[a, b]$ then $|f| \le M$. Then due to (11)

$$F(t) - F(c) = \int_a^t f(x)\,dx - \int_a^c f(x)\,dx = \int_c^t f(x)\,dx \tag{14}$$

and

$$|F(t) - F(c)| = |\int_c^t f(x)\,dx| \le M|\int_c^t dx| = M|t - c|$$

and this implies continuity: $|t - c| \le \delta = \epsilon/M \implies |F(t) - F(c)| \le \epsilon$.

(ii) Further, if f is continuos at c then rewriting (14) as

$$F(t)-F(c)==\int_c^t f(c)\,dx+\int_c^t \bigl(f(x)-f(c)\bigr)\,dx=$$
$$f(c)(t-c)+\int_c^t \bigl(f(x)-f(c)\bigr)\,dx$$

we conclude that

$$|F(t)-F(c)-f(c)(t-c)|\le |\int_c^t |f(x)-f(c)|\,dx|\le \epsilon|\int_c^t dx|=\epsilon|t-c|$$

which implies that $F'(c)=f(c)$. ☐

Antiderivative and Newton-Leibnitz formula

So, we proved that as f is contionuous, then

$$\Bigl(\int_a^t f(x)\,dx\Bigr)'=f(x). \tag{15}$$

This implies the first statement of

Theorem 13. *Let f be continuous. Then*

(i) Function $F(t)$ defined by (13) is antiderivative of f: $F'=f$.

(ii) Further, antiderivative is defined modulo additive constant: F_1 is also antiderivative of f iff $F_1(t)-F(t)=C(=\text{const})$.

Proof. We need to prove (ii). However $F_1'=f$ iff $(F_1-F)'=0$ and this holds iff $F-F_1=$ const (we proved it as a corollary of Lagrange theorem). ☐

Theorem 14. *(Newton-Leibnitz formula) Let f be continuous and F be antiderivative of f. Then*

$$\int_a^b f(x)\,dx=F(x)\big|_a^b=F(b)-F(a). \tag{16}$$

Proof. Since $\int_a^t f(x)\,dx$ is also antiderivative of f, we conclude that $\int_a^t f(x)\,dx=F(t)+C$. Plugging $t=a$ we conclude that $F(a)=-C$ and therefore $\int_a^t f(x)\,dx=F(t)-F(a)$. Plugging $t=b$ we get (16). ☐

So, we got an effective tool to calculate integrals.