Chapter 5

5.2 Integral of continuous functions

Our ultimate goal is to define

$$\int_a^b f(x)\,dx$$

with $a < b$ and function f defined on $[a, b]$ (under some assumptions).

Riemannian sums

First of all, consider *partition* $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a, b]$:

$$I = [a, b] = I_1 \cup I_2 \cup \cdots \cup I_n, \qquad I_j = [x_{j-1}, x_j]$$

and consider points $\xi_j \in I_j$ (so $x_{j-1} \le \xi_j \le x_j$. Let us introduce *Riemannian Sum*

$$\text{RS}(I, f; \mathcal{P}, \xi_1, \dots, \xi_n) = \sum_{j=1}^{n} f(\xi_j) \cdot \Delta_j \tag{1}$$

where $\Delta_j = x_j - x_{j-1}$ is the length of I_j.

Riemannian sum is an approximation to integral if the latter is defined but it depends not only on a, b, f but also on the choice of partition and points $\xi_1, \dots, \xi_n$.

Among all the Riemannian sums with the given partition there are the largest and the smallest ones: *Upper Riemannian Sum*

$$\text{URS}(I, f; \mathcal{P}) = \sum_{j=1}^{n} \max_{\xi \in I_j} f(\xi) \cdot \Delta_j \tag{2}$$

and *Lower Riemannian Sum*

$$\text{LRS}(I, f; \mathcal{P}) = \sum_{j=1}^{n} \min_{\xi \in I_j} f(\xi) \cdot \Delta_j. \tag{3}$$

Remark 1. (i) If f is not continuous, $\max_{\xi \in I_j}$ and $\min_{\xi \in I_j}$ do not necessarily exist and one should replace them by $\sup_{\xi \in I_j}$ and $\inf_{\xi \in I_j}$ respectively;

(ii) If $\sup \xi \in I f(\xi) = +\infty$ then $\sup \xi \in I_j f(\xi) = +\infty$ for some j and then $\text{URS}(I, f; \mathcal{P}) = +\infty$. Similarly, if $\inf \xi \in I f(\xi) = -\infty$ then $\inf \xi \in I_j f(\xi) = -\infty$ for some j and then $\text{LRS}(I, f; \mathcal{P}) = -\infty$. In these cases integral will not be defined.

Condition 2. So, let us assume that f is bounded on $[a,b]$: $|f(x)| \le M$.

Obviously,

$$\operatorname{LRS}(I,f;\mathcal{P}) \le \operatorname{RS}(I,f;\mathcal{P},\xi_1,\dots,\xi_n) \le \operatorname{URS}(I,f;\mathcal{P}). \tag{4}$$

Proposition 3. *If $\mathcal{P} \subset \mathcal{P}'$ then*

$$\begin{aligned}\operatorname{URS}(I,f;\mathcal{P}') &\le \operatorname{URS}(I,f;\mathcal{P}),\\ \operatorname{LRS}(I,f;\mathcal{P}') &\le \operatorname{LRS}(I,f;\mathcal{P}).\end{aligned}$$

Proof. Consider only first inequality. Assume that $\mathcal{P}'$ differ from $\mathcal{P}$ by just one added point x'_j between x_{j-1} and x_j. Then $\operatorname{URS}(I,f;\mathcal{P}')$ contains

$$\max_{x_{j-1}\le\xi\le x'_j} f(\xi)\cdot(x'_j - x_{j-1}) + \max_{x'_j\le\xi\le x'_j} f(\xi)\cdot(x_j - x'_j) \tag{5}$$

instead of

$$\max_{x_{j-1}\le\xi\le x_j} f(\xi)\cdot(x_j - x_{j-1}). \tag{6}$$

However, since $\max_{x_{j-1}\le\xi\le x'_j} f(\xi) \le \max_{x_{j-1}\le\xi\le x_j} f(\xi)$
and $\max_{x'_j\le\xi\le x'_j} f(\xi) \le \max_{x_{j-1}\le\xi\le x_j} f(\xi)$ (one of these inequalities is an equality), expression (**??**) does not exceed (**??**).

Now adding point after point to partition we conclude the proof. $\square$

Proposition 4. *Any Upper Riemannian Sum is greater than or equal to any Lower Riemannian Sum*

$$\operatorname{LRS}(I,f;\mathcal{P}') \le \operatorname{URS}(I,f;\mathcal{P}).$$

Proof. Consider partition $\mathcal{P}\cup\mathcal{P}'$ (so we swap points from both). Then

$$\operatorname{LRS}(I,f;\mathcal{P}') \le \operatorname{LRS}(I,f;\mathcal{P}'\cup\mathcal{P}) \le \operatorname{URS}(I,f;\mathcal{P}'\cup\mathcal{P}) \le \operatorname{URS}(I,f;\mathcal{P})$$

where middle inequality follows from (**??**) and two others from proposition **??**. $\square$

Definition of integral

Definition 5. If there exists unique number S such that

$$\mathrm{LRS}(I,f;\mathcal{P}') \le S \le \mathrm{URS}(I,f;\mathcal{P}).$$

for all partitions $\mathcal{P},\mathcal{P}'$, then S is an integral of f from a to b:

$$S = \int_a^b f(x)\,dx;$$

then f is *integrable* on $[a,b]$.

Remark 6. Assumption that S is unique means exactly that

$$\sup_{\mathcal{P}'} \mathrm{LRS}(I,f;\mathcal{P}') = S = \inf_{\mathcal{P}} \mathrm{URS}(I,f;\mathcal{P}). \tag{7}$$

Otherwise $S=\int_a^b f(x)\,dx$ is not defined and then f is *not integrable* on $[a,b]$.

When an integral exists?

Theorem 7. $S=\int_a^b f(x)\,dx$ *iff*

$$\forall\epsilon>0\exists\delta>0 \max_j \Delta_j \le \delta \implies |\mathrm{RS}(I,f;\mathcal{P},\xi_1,\ldots,\xi_n) - S| \le \epsilon. \tag{8}$$

Proof. If (**??**) holds then it holds for $\mathrm{URS}(I,f;\mathcal{P})$ and $\mathrm{LRS}(I,f;\mathcal{P})$ as well and then $\mathrm{URS}(I,f;\mathcal{P}) \le S+\epsilon$ and since it holds for any $\epsilon>0$ as $\max_j \Delta_j < \delta$ we conclude that $\inf_{\mathcal{P}} \mathrm{URS}(I,f;\mathcal{P}) \le S$.

Similarly $\sup_{\mathcal{P}} \mathrm{URS}(I,f;\mathcal{P}) \ge S$. Since $\sup_{\mathcal{P}} \mathrm{URS}(I,f;\mathcal{P}) \le \inf_{\mathcal{P}} \mathrm{URS}(I,f;\mathcal{P})$ we conclude that $\sup_{\mathcal{P}} \mathrm{URS}(I,f;\mathcal{P}) = S = \inf_{\mathcal{P}} \mathrm{URS}(I,f;\mathcal{P})$.

Conversely: assume that $\sup_{\mathcal{P}} \mathrm{URS}(I,f;\mathcal{P}) = S = \inf_{\mathcal{P}} \mathrm{URS}(I,f;\mathcal{P})$. Then for any $\epsilon>0$ there exist partitions $\bar{\mathcal{P}}$ and $\bar{\mathcal{P}}'$ such that $\mathrm{URS}(I,f;\bar{\mathcal{P}}) \le S+\frac{\epsilon}{2}$ and $\mathrm{LRS}(I,f;\bar{\mathcal{P}}') \ge S-\frac{\epsilon}{2}$. However theorem says that the similarinequalities must be true for any partition $\mathcal{P}$ with $\max_j \Delta_j \le \delta$.

Let us prove the first one. Fix $\bar{\mathcal{P}} = \{\bar{x}_0,\ldots\bar{x}_m\}$ with m points and consider $\bar{\mathcal{P}}\cup\mathcal{P}$. Then $\mathrm{URS}(I,f;\bar{\mathcal{P}}\cup\mathcal{P}) \le \mathrm{URS}(I,f;\bar{\mathcal{P}}) \le S+\epsilon$. Compare $\mathrm{URS}(I,f;\bar{\mathcal{P}}\cup\mathcal{P})$ and $\mathrm{URS}(I,f;\mathcal{P})$; the latter is larger but not much. Really, the only difference comes from no more than $(m-1)$ intervals from partition $\mathcal{P}$ covering points $\bar{x}_1,\ \ldots,\ \bar{x}_{m-1}$. Contribution of each does not exceed $2M\delta$ where M is an upper bound for $|f(x)|$. Then the total difference does not exceed $2M(m-1)\delta$. Now select $\delta>0$ so small that $2M(m-1)\delta < \frac{\epsilon}{2}$.

Then

$$\mathrm{URS}(I,f;\mathcal{P}) \le \mathrm{URS}(I,f;\bar{\mathcal{P}}\cup\mathcal{P}) + \frac{\epsilon}{2} \le S+\frac{\epsilon}{2}+\frac{\epsilon}{2} = S+\epsilon.$$

Siilarly $\mathrm{LRS}(I, f; \mathcal{P}) \geq S - \epsilon$ and then $\mathrm{RS}(I, f; \mathcal{P}, \xi_1, \ldots, \xi_n)$ is between $\mathrm{LRS}(I, f; \mathcal{P}) \geq S - \epsilon$ and $\mathrm{URS}(I, f; \mathcal{P}) \geq S + \epsilon$, which is exactly (**??**). ☐

Then we conclude immediately that

Theorem 8. *f is integrable on $[a, b]$ iff*

$$\forall \epsilon > 0 \exists \delta > 0 \max_j \Delta_j \leq \delta \implies |\mathrm{URS}(I, f; \mathcal{P}) - \mathrm{LRS}(I, f; \mathcal{P})| \leq \epsilon. \tag{9}$$

Note that

$$\mathrm{URS}(I, f; \mathcal{P}) - \mathrm{LRS}(I, f; \mathcal{P}) = \sum_{j=1}^{n} \mathrm{osc}_{I_j} f \cdot \Delta_j \tag{10}$$

where

$$\mathrm{osc}_J f = \max_J f - \min_J f \tag{11}$$

is *oscillation* of f on J.

Theorem 9. *If f is continuous on $[a, b]$ then f is integrable on $[a, b]$.*

Proof. We know that if f is continuous on $[a, b]$ then f is uniformly continuous: for any $\epsilon > 0$ there exists $\delta > 0$ such that $|x' - x| \leq \delta$ implies that $|f(x') - f(x)| \leq \epsilon$. So, $\Delta_j \leq \delta \implies \mathrm{osc}_{I_j} f \leq \epsilon$ and

$$0 \leq \mathrm{URS}(I, f; \mathcal{P}) - \mathrm{LRS}(I, f; \mathcal{P}) = \sum_{j=1}^{n} \mathrm{osc}_{I_j} f \cdot \Delta_j \leq \sum_{j=1}^{n} \epsilon \cdot \Delta_j = \epsilon(b - a).$$

This and theorem **??** complete the proof. ☐