Chapter 3

3.9 Differential. Newton Method

Differential

First of all, from it follows from the definition of derivative that

$$\frac{f(x') - f(x)}{x' - x} = f'(x) + \epsilon(x', x)$$

with $\epsilon \to 0$ as $x' \to x$; therefore

$$f(x') - f(x) = f'(x)(x' - x) + \epsilon(x', x)(x' - x)$$

or denoting $\Delta x = x' - x$, $\Delta f = f(x') - f(x) = f(x + \delta x) - f(x)$ we arrive to

$$\Delta f = f'(x)\Delta x + \epsilon(x, \Delta x)\Delta x, \qquad \epsilon \to 0 \quad \text{as } \Delta x \to 0. \tag{1}$$

Definition 1. We call Δf and Δx *increment* of f, x respectively.

Definition 2. We call $df = f'(x)\Delta x$ *differential* of f. In particular, as $f(x) = x$ we get $dx = \Delta x$. So, $df = f'(x)\,dx$.

Notion of differential has ample generalizations. Note that df is linear with respect df and that equation $y = f(x_0) + f'(x_0)(x - x_0)$ defines a straight line tangent at x_0 to the curve $y = f(x)$.

Theorem 3. *Definition of the differential remains valid if argument of the function is a function as well.*

Proof. Due to chain rule

$$d(f \circ g) = (f \circ g)' dx = (f' \circ g) \times g' dx = (f' \circ g) dg. \tag{2}$$

□

I want to better estimate an error $\epsilon\Delta x$ in (1).

Theorem 4. *If f is twice differentiable between x and $x + \Delta x$ and $|f''| \le M$ there then*

$$|\Delta f - df| \le \frac{M}{2}|\Delta x|^2. \tag{3}$$

Proof. Consider $F(x) = f(x) - f(a) - f'(x)(x-a)$ and $G(x) = (x-a)^2$. Then by Cauchy MVT

$$\frac{F(b)-F(a)}{G(b)-G(a)} = \frac{F'(x)}{G'(x)}$$

where x is a point between a and b (we assume that $a \neq b$ rather than $a < b$). Note that $F(a = G(a) = 0$ and $G'(x) = 2(x-a)$, $F'(x) = f'(x) - f''(x)\cdot(x-a) - f'(x)\cdot 1 = -f''(x)(x-a)$ and therefore

$$\frac{F(b)}{(b-a)^2} = -\frac{f''(x)(x-a)}{2(x-a))} = -\frac{1}{2}f''(t)$$

where t is some point between a and b. Then $F(b) = -\frac{1}{2}f''(x)(b-a)^2$ or $f(b) - f(a) - f'(b)(b-a) = -\frac{1}{2}f''(x)(b-a)^2$. Plugging $b = x$ and $a = x + \Delta x$ we arrive to

$$f(x+\Delta x) - f(x) - f'(x)\Delta x = \frac{1}{2}f''(t)(\Delta x)^2 \tag{4}$$

where t is between x and $x + \Delta x$. This implies (3). □

Example 1. Calculate $\sqrt{1.1}$.
Consider $f(x) = x^{1/2}$ Then $f'(x) = \frac{1}{2}x^{-1/2}$, $f''(x) = -\frac{1}{4}x^{-3/2}$.
Plugging $x = 1$, $\Delta x = 0.1$ we see that $|f''| \le M = \frac{1}{4}$ as $1 \le x \le 1.1$ and therefore with an error not exceeding $\frac{1}{8}(0.1)^2 = 0.000125$ $\sqrt{1.1} \approx 1 + \frac{1}{2}\cdot 0.1 = 1.05$.

Newton Method

Newton method allows to find approximately solution of equation $f(x) = 0$ (under certain conditions).

> Consider x_0 "a candidate" to solutions. Then replacing equation $f(x) = 0$ by an approximate equation $f(x_0) + f'(x_0)(x - x_0) = 0$ and solving it $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ we obtain a new candidate to solutions. Repeating this process
>
> $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{5}$$
>
> we hope to get a better and better approximation to solution.

Thus Newton method is *iterative*: solution obtained on one step is used on the next one as a starting point.

> **Theorem 5.** *Let f be defined on (a,b) and twice differentiable there. Assume that $f(a) < 0$, $f(b) > 0$ and $f'(x) \ge m$ with some $m > 0$ on (a,b). Let $|f''| \le M$ on (a,b). Then if $|f(x_0)| = \epsilon$ then x_n approximate solution with an error not exceeding*
>
> $$|x_n - c| \le \frac{2m}{M}\Bigl(\frac{M\epsilon}{2m^2}\Bigr)^{2^n} \tag{6}$$
>
> *(as long as all x_n belong to (a,b) which is the case provided either $f'' \ge 0$ on (a,b) and $f(x_0) > 0$ or ϵ is small enough.*

Proof. Let c be (an unknown) root $f(c) = 0$. In frames of the theorem there is exactly one root on (a, b). Then

$$|x_0 - c| \le \epsilon/m. \tag{7}$$

It follows from $|f(x_0) - f(c)| \ge m|x - c|$. Now

$$x_{n+1} - c = x_n - c - \frac{f(x_n)}{f'(x_n)} = -\frac{f(x_n) - f'(x_n)(x_n - c)}{f'(x_n)} = -\frac{f(x_n) - f'(x_n)(x_n - c) - f(c)}{f'(x_n)}$$

since $f(c) = 0$. Then the numerator does not exceed $M(x_n - c)^2/2$ and denominator is no less than m. So if $\Delta_n = |x_n - c|$ we arrive to $\Delta_{n+1} \le \theta \Delta_n^2$ with $\theta = \dfrac{M}{2m}$. Equivalently $(\theta\Delta_{n+1}) \le (\theta\Delta_n)^2$.

Then one can prove easily by induction that $\theta\Delta_n \le (\theta\Delta_0)^{2^n}$. Plugging θ and $\Delta_0 \le \epsilon/m$ we arrive to $\theta\Delta_0 \le \frac{M\epsilon}{2m^2}$ which implies (6). □