Chapter 10. L'Hôpital rule

10.4 L'Hôpital rule

L'Hôpital rule is used to calculate limits when indefinities such as $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ appear.

Cauchy MVT

Theorem 1. *Let f and g be differentiable on (a,b). Then there exists $c \in (a,b)$ such that*

$$\big(f(b) - f(a)\big)g'(c) = \big(g(b) - g(a)\big)f'(c). \tag{1}$$

Proof. Consider function

$$G(x) = \big(f(x) - f(a)\big)\big(g(b) - g(a)\big) - \big(f(b) - f(a)\big)\big(g(x) - g(a)\big).$$

As $x = a$ both "blue" factor vanish and $G(a) = 0$. As $x = b$ "blue" factors are equal to $\big(f(b) - f(a)\big)$ and $\big(g(b) - g(a)\big)$ respectively and both terms compensate one another so $G(b) = 0$. Therefore due to the Rolle's theorem $G'(c) = 0$ for some $c \in (a,b)$.
However

$$G'(x) = f'(x)\big(g(b) - g(a)\big) - \big(f(b) - f(a)\big)g'(x)$$

which implies (1). □

L'Hôpital rule. I.

Consider $\lim_{x\to c^\pm} \frac{f(x)}{g(x)}$ assuming that f, g are differentiable and $f(a) = g(a) = 0$. Later we consider $x \to \pm\infty$ and the case when $f(x) \to \infty$, $g(x) \to \infty$.

Theorem 2. *Let $f(x)$ and $g(x)$ be differentiable on (a,b) and $f(c) = g(c) = 0$. Assume that*

$$\lim_{x\to c^\pm} \frac{f'(x)}{g'(x)} \tag{2}$$

exists. Then $\displaystyle\lim_{x\to c^\pm} \frac{f(x)}{g(x)}$ also exists and both limits are equal.

Proof. Since $f(c) = g(c) = 0$

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(c)}{g(x) - g(c)}$$

and due do Cauchy MVT

$$\big(f(x) - f(c)\big)g'(\xi) = \big(g(x) - g(c)\big)f'(\xi). \tag{3}$$

Since we assume that (2) exists $g'(\xi) \neq 0$ as $\xi \in (c-\delta, c) \cup (c, c+\delta)$. But then $g(x) \neq 0$ as well: due to MVT $g(x) = g(x) - g(c) = g'(\eta)(x-c)$ with η also between c and x and $g'(\eta) \neq 0$. So, dividing by $g'(\xi)$, $g(x)$ we get

$$\frac{f(x)}{g(x)} = \frac{f'(\xi)}{g'(\xi)}.$$

As $x \to c^{\pm}$ also $\xi \to c^{\pm}$ and therefore if finite or infinite limit (2) exists, then finite or infinite limit $\lim\limits_{x \to c^{\pm}} \dfrac{f(x)}{g(x)}$ also exists and both limits are equal. □

Example 1. Converse statement is not true, Really, consider

$$\lim_{x \to 0} \frac{x^2 \cdot \sin\left(\frac{1}{x}\right)}{x} = \lim_{x \to 0} x \cdot \sin\left(\frac{1}{x}\right) = 0.$$

On the other hand applying L'Hôpital rule we would get

$$\lim_{x \to 0} \frac{\left(x^2 \cdot \sin\left(\frac{1}{x}\right)\right)'}{1} = 2x \cdot \sin\left(\frac{1}{x}\right) + x^2 \cdot \cos\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right) = 2x \cdot \sin\left(\frac{1}{x}\right) + -\cos\left(\frac{1}{x}\right)$$

and while the first term tends to 0, the second has no limit.
So, $\lim_{x \to c^{\pm}} \frac{f(x)}{g(x)}$ exists but $\lim_{x \to c^{\pm}} \frac{f'(x)}{g'(x)}$ does not.

L'Hôpital rule. II.

Theorem 3. *Let $f(x)$ and $g(x)$ be differentiable on (a, b) and $\lim_{x \to c^{\pm}} f(x) = \lim_{x \to c^{\pm}} g(x) = \infty$. Assume that*

$$\lim_{x \to c^{\pm}} \frac{f'(x)}{g'(x)}$$

exists. Then $\lim\limits_{x \to c^{\pm}} \dfrac{f(x)}{g(x)}$ also exists and both limits are equal.

Proof. I will start with pseudo-proof:
Since $\lim\limits_{x \to c^{\pm}} \dfrac{1}{f} = \lim\limits_{x \to c^{\pm}} \dfrac{1}{g} = 0$ one can apply Theorem 2

$$\lim_{x \to c^{\pm}} \frac{g}{f} = \lim_{x \to c^{\pm}} \frac{\frac{1}{f}}{\frac{1}{g}} = \lim_{x \to c^{\pm}} \frac{\left(\frac{1}{f}\right)'}{\left(\frac{1}{g}\right)'} = \lim_{x \to c^{\pm}} \frac{-\frac{f'}{f^2}}{-\frac{g'}{g^2}} = \lim_{x \to c^{\pm}} \left(\frac{f'}{g'} \cdot \frac{g^2}{f^2}\right) = \left(\lim_{x \to c^{\pm}} \frac{f'}{g'}\right) \times \left(\lim_{x \to c^{\pm}} \frac{g}{f}\right)^2$$

which implies equality of limits. What is wrong with this proof?

It assumes that $\lim\limits_{x \to c^{\pm}} \dfrac{f(x)}{g(x)}$ exists and is neither 0 nor ∞.

So, let us give a real proof. Consider $x \to c^{+}$ for simplicity of notations. Consider

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(t)}{g(x) - g(t)} - \frac{f(x)}{g(x)} \cdot \frac{g(t)}{g(x) - g(t)} + \frac{f(t)}{g(x) - g(t)}$$

where we substracted $f(t)$ and $g(t)$ to apply Cauchy MVT and the rest is the correction. Then

$$\frac{f(x)}{g(x)}\Big(1+\frac{g(t)}{g(x)-g(t)}\Big) = \frac{f(x)-f(t)}{g(x)-g(t)} + \frac{f(t)}{g(x)-g(t)}$$

and finally

$$\frac{f(x)}{g(x)} = \Big(1+\frac{g(t)}{g(x)-g(t)}\Big)^{-1}\Big(\frac{f(x)-f(t)}{g(x)-g(t)} + \frac{f(t)}{g(x)-g(t)}\Big).$$

By Theorem 1 or $c<x<t$ we can replace $\dfrac{f(x)-f(t)}{g(x)-g(t)}$ by $\dfrac{f'(\xi)}{g'(\xi)}$ with $x<\xi<t$.

Now fix t such that $|\dfrac{f'(\xi)}{g'(\xi)} - L| < \epsilon/9$ if $\lim_{x\to c^\pm}\frac{f'(x)}{g'(x)} = L$ is finite or $\dfrac{f'(\xi)}{g'(\xi)} \geq 9M$ if this limit was $+\infty$ (and make similar arrangements if limit was $-\infty$ or just ∞).

Now, if L is finite, after ϵ and consequently t are chosen, we can choose $\delta>0$ such that $|\dfrac{g(t)}{g(x)-g(t)}| < \dfrac{\epsilon}{9(|L|+1)}$ and $|\dfrac{f(t)}{g(x)-g(t)}| < \dfrac{\epsilon}{9}$ as $c<x<c+\delta$ where we use that $\lim g(x)=\infty$. Then $|\dfrac{f(x)}{g(x)} - L| < \epsilon$.

Similarly, if L is infinite, after M and consequently t are chosen, we can choose $\delta>0$ such that $|\dfrac{g(t)}{g(x)-g(t)}| < \dfrac{1}{9}$ and $|\dfrac{f(t)}{g(x)-g(t)}| < \dfrac{1}{9}$ as $c<x<c+\delta$ where we use that $\lim g(x)=\infty$. Then $\dfrac{f(x)}{g(x)} \geq M$. □

L'Hôpital rule. Full Glory.

Finally, we need to extend L'Hôpital rule to the case when $x\to\pm\infty$ or $x\to\infty$.

Theorem 4. *Let $f(x)$ and $g(x)$ be differentiable on (M,∞) or $(-\infty,-M)$ and either*

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} g(x) = \infty$$

or

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} g(x) = 0.$$

Assume that

$$\lim_{x\to\pm\infty}\frac{f'(x)}{g'(x)}$$

exists. Then $\displaystyle\lim_{x\to\pm\infty}\frac{f(x)}{g(x)}$ also exists and both limits are equal.

Proof. Consider $F(t) = f(1/t)$, $G(t) = g(1/t)$. Then

$$\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = \lim_{t\to\pm 0} \frac{F(t)}{G(t)} == \lim_{t\to\pm 0} \frac{F'(t)}{G'(t)} \overset{\boxed{\text{by Theorem 2 or 3}}}{=}$$

$$\lim_{t\to\pm 0} \frac{f'(\frac{1}{t}) \cdot (-\frac{1}{t^2})}{g'(\frac{1}{t}) \cdot (-\frac{1}{t^2})} = \lim_{t\to\pm 0} \frac{f'(\frac{1}{t})}{g'(\frac{1}{t})} = \lim_{x\to\pm\infty} \frac{f'(x)}{g'(x)}.$$

□