Chapter 4. Mean Value Theorem and Applications

4.4 Convexity and concativity

Geometry

Definition 1. Let $\mathcal{D}$ be a planar figure and Γ its boundary. Then

(i) $\mathcal{D}$ is *convex* if for all $A, B \in \mathcal{D} \cup \Gamma$ the whole segment $(A, B) \subset \mathcal{D} \cup \Gamma$.

(ii) $\mathcal{D}$ is *strictly convex* if for all $A, B \in \mathcal{D} \cup \Gamma$ the whole segment $(A, B) \subset \mathcal{D}$.

Example 1. Disk is strictly convex. Rectangle is convex but not strictly convex. If Γ contains a straight segment then $\mathcal{D}$ cannot be strictly convex.

Functions

Definition 2. Let $f(x)$ be defined on (a, b). Then

(i) Let (x) is called convex if its *overgraph* $\{(x, y) :\ x \in (a, b), y \geq f(x)\}$ is convex.

(ii) Let (x) is called concave if its *undergraph* $\{(x, y) :\ x \in (a, b), y \leq f(x)\}$ is convex.

Since boundary of overgraph and undergraph contains straight segments, they cannot be strictly convex. So, we will wait a bit with the corresponding definition.
One can see easily that

Theorem 3. *(i) Function $f(x)$ defined on (a, b) is convex iff for all $x, x' : a < x < x' < b$ a straigt segment $\big((x, f(x)), (x', f(x'))\big)$ is above the graph of the function.*

(ii) Function $f(x)$ defined on (a, b) is concave iff for all $x, x' : a < x < x' < b$ a straigt segment $\big((x, f(x)), (x', f(x'))\big)$ is below the graph of the function.

Now we can define strictly convex and concave functions:

Definition 4. (i) Function $f(x)$ defined on (a, b) is *strictly convex* iff for all $x, x' : a < x < x' < b$ a straigt segment $\big((x, f(x)), (x', f(x'))\big)$ is strictly above the graph of the function.

(ii) Function $f(x)$ defined on (a, b) is *strictly concave* iff for all $x, x' : a < x < x' < b$ a straigt segment $\big((x, f(x)), (x', f(x'))\big)$ is strictly below the graph of the function.

The following theorem follows directly from definitions:

Theorem 5. *(i) Function $f(x)$ defined on (a,b) is convex iff*

$$f\big((1-t)x+tx'\big) \le (1-t)f(x)+tf(x') \qquad \forall x,x' : a<x<x'<b \ \forall t\in(0,1);$$

(ii) Function $f(x)$ defined on (a,b) is strictly convex iff

$$f\big((1-t)x+tx'\big) < (1-t)f(x)+tf(x') \qquad \forall x,x' : a<x<x'<b \ \forall t\in(0,1);$$

(iii) Function $f(x)$ defined on (a,b) is concave iff

$$f\big((1-t)x+tx'\big) \ge (1-t)f(x)+tf(x') \qquad \forall x,x' : a<x<x'<b \ \forall t\in(0,1);$$

(iv) Function $f(x)$ defined on (a,b) is strictly concave iff

$$f\big((1-t)x+tx'\big) > (1-t)f(x)+tf(x') \qquad \forall x,x' : a<x<x'<b \ \forall t\in(0,1).$$

Theorem 6. *Convex/concave function $f(x)$ defined on (a,b) is continuous.*

Proof. Let f be convex. Consider $c\in(a,b)$. Then for $x<c$ $f(x)\le f(c)(b-x)(b-c)^{-1}+f(b)(x-c)(b-c)^{-1}$ and therefore $f(x)-f(c)\le (f(b)-f(c))(x-c)(b-c)^{-1}$ and for all $\epsilon>0$ there exists $\delta>0$ such that if $x\in(c,c+\delta)$ then $f(x)\le f(c)+\epsilon$. The same is true as $x\in(c-\delta,c)$.
On the other hand, considering $x'=2c-x$ and $t=\frac{1}{2}$ we conclude that $f(c)\le\frac{1}{2}\big(f(x)+f(2x-c)\big)$ as $|x-c|\le\delta$; but then $f(2x-c)\le f(c)+\epsilon$ and $2f(c)\le f(x)+f(c)+\epsilon$ which implies $f(x)\ge f(c)-\epsilon$. So, $|f(x)-f(c)|\le\epsilon$ as $|x-c|<\delta$. □

Theorem 7. *Strictly convex function has only one local minimum. Strictly concave function has only one local maximum.*

Proof. Consider convex functions only. Assume that $c<d$ be two local minima. Consider

$$k=\frac{f(d)-f(c)}{d-c}$$

a slope of the segment $\big(c,f(c)),(d,f(d))\big)$. If $k>0$ then for $x\in(c,d)$ $f(x)\le f(d)+k(x-d)<f(d)$ and d cannot be a local minimum. If $k<0$ then for $x\in(c,d)$ $f(x)\le f(c)+k(x-c<f(d)$ and c cannot be a local minimum. So, $k=0$ but then $x\in(c,d)$ $f(x)<f(c)$ (this is the only moment we use *strict* convexity. Contradiction. □

First derivative criteria

Theorem 8. *If function f is differentiable and f' increases the f is strictly convex. If f is differentiable and f' decreases the f is strictly concave.*

Proof. Consider $a \le c < d \le b$ and $x = (1-t)c + td$, $t \in (0,1)$. By mean value theorem $\dfrac{f(x)-f(c)}{x-c} = f'(\xi)$ and therefore $f(x) = f(c) + f'(\xi)(x-c)$ with $\xi \in (c,x)$. Similarly $f(x) = f(d) + f'(\eta)(x-d)$ with $\eta \in (x,d)$. Therefore

$$\begin{aligned} f(x) = (1-t)f(x) + tf(x) = \\ &(1-t)f(c) + tf(d) + f'(\xi)(x-c)(1-t) + f'(\eta)(x-d)t = \\ &(1-t)f(c) + tf(d) + (d-c)t(1-t)\big(f'(\xi) - f'(\eta)\big) < (1-t)f(c) + tf(d) \end{aligned}$$

since $\xi < x < \eta$ and therefore $f'(\xi) < f'(\eta)$ assuming f' is increasing. So $f\big((1-t)c+td\big) < (1-t)f(c) + tf(d)$ which means a strict convexity.
If f' is increasing we would get $f\big((1-t)c+td\big) > (1-t)f(c) + tf(d)$ which means a strict concativity. $\square$

Remark 9. Replacing *increases* by *not decreases* or *decreases* by *not increases* we would get convex or concave (but not necessarily strictly).

In the textbook convexity/concativity is defined by "f' increases" and "f' decreases". This eliminates piecewise differentiable functions like $|x|$ from consideration.

Second derivative criteria

Theorem 10. *If f is twice differentiable and $f'' \ge 0$ then f is convex. If f is twice differentiable and $f'' > 0$ then f is strictly convex. If f is twice differentiable and $f'' \le 0$ then f is concave. If f is twice differentiable and $f'' < 0$ then f is strictly concave.*

Proof. Trivially from the first derivative criteria of increasing/decreasing function. $\square$

Inflection points

Definition 11. Pint $c \in (a,b)$ is an *inflection point* of f iff either
- f is strictly convex on $(c-\delta, c)$ and strictly concave on $(c, c+\delta)$ or
- f is strictly concave on $(c-\delta, c)$ and strictly convex on $(c, c+\delta)$.

The following theorem follows immediately:

Theorem 12. *If f is twice differentiable on (a,b) and either*
- $f'' < 0$ on $(c-\delta, c)$ and $f'' > 0$ on $(c, c+\delta)$ or
- $f'' > 0$ on $(c-\delta, c)$ and $f'' < 0$ on $(c, c+\delta)$.
Then c is an inflection point.

Third derivative criteria

Theorem 13. *Let f be trice differentiable, $f''(c) =$ and $f'''(c) \neq 0$. Then c is an inflection point.*

Proof. Follows from the previous theorem. ☐

Higher derivative criteria

Theorem 14. *Let f by n times differentiable,*

$$f''(c) = f'''(c) = \cdots = f^{(n-1)}(c) = 0, \qquad f^{(n)}(c) \neq 0.$$

Then if n is odd, c is an inflection point.
If n is even then f is strictly convex/concave on $(c-\delta, c+\delta)$ as $f^{(n)}(c) > 0$ / $f^{(n)}(c) < 0$ respectively.

Proof. One needs just investigate the sign of f'' exactly as we did in higher derivative criteria for maximum/minimum. ☐

Example 2. Consider $x + x^2$, $x - x^2$, $x + x^3$, $x - x^3$, $x + x^4$, $x - x^4$, $x + x^5$, $x - x^5$.