

Chapter 4. Mean Value Theorem and Applications

4.3 Local extremums at boundary points

Monotonicity criteria

Theorem 1. Let $f(x)$ be defined on $[a, a + \delta)$ and continuous at a .

(i) Let f be decreasing on $(a, a + \delta)$. Then a is a local maximum, this maximum is strict and isolated (which means that a is the only extremum on $[a, a + \delta)$).

(ii) Let f be increasing on $(a, a + \delta)$. Then a is a local minimum, this minimum is strict and isolated (which means that a is the only extremum on $[a, a + \delta)$).

On the other hand, let $f(x)$ be defined on $(b - \delta, b]$ and continuous at b .

(iii) Let f be decreasing on $(b - \delta, b)$. Then b is a local minimum, this minimum is strict and isolated.

(iv) Let f be increasing on $(b - \delta, b)$. Then b is a local maximum, this maximum is strict and isolated.

Proof. The same arguments as in inner points work. □

First derivative criteria

Theorem 2. Let $f(x)$ be defined on $[a, a + \delta)$ and continuous at a . Let f be differentiable on $(a, a + \delta)$.

(i) Let $f' < 0$ on $(a, a + \delta)$. Then a is a local maximum, this maximum is strict and isolated.

(ii) Let $f' > 0$ on $(a, a + \delta)$. Then a is a local minimum, this minimum is strict and isolated.

On the other hand, let $f(x)$ be defined on $(b - \delta, b]$ and continuous at b .

(iii) Let $f' < 0$ on $(b - \delta, b)$. Then b is a local minimum, this minimum is strict and isolated.

(iv) Let $f' > 0$ on $(b - \delta, b)$. Then b is a local maximum, this maximum is strict and isolated.

Proof. Follows from theorem 1 and the theorem that $f' > 0$ implies f increasing and $f' < 0$ implies f decreasing. □

Example 1. Consider x, x^2, x^3, x^4 on $(-\infty, 0]$ and on $[0, \infty)$. Look at textbook examples.

Higher derivative criteria

Theorem 3. Let $f(x)$ be defined and differentiable up to order $(n-1)$ (including) on $[a, a + \delta)$, $f'(a) = f''(a) = \dots = f^{(n-1)}(a) = 0$. Moreover, let $f^{(n)}(a)$ exist and differ from 0.

(i) Let $f^{(n)}(a) < 0$. Then a is a local maximum, this maximum is strict and isolated.

(ii) Let $f^{(n)}(a) > 0$. Then a is a local minimum, this minimum is strict and isolated.

Let $f(x)$ be defined and differentiable up to order $(n-1)$ (including) on $(b-\delta, b]$, $f'(b) = f''(b) = \dots = f^{(n-1)}(b) = 0$. Moreover, let $f^{(n)}(b)$ exist and differ from 0.

(i) Let $(-1)^n f^{(n)}(b) < 0$. Then b is a local maximum, this maximum is strict and isolated.

(ii) Let $(-1)^n f^{(n)}(b) > 0$. Then b is a local minimum, this minimum is strict and isolated.

Proof. The same ε

□

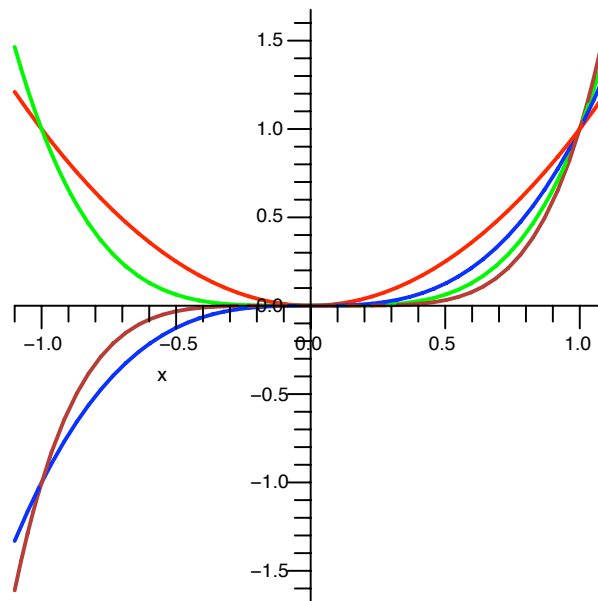


Figure 1: Again graphs of function $f(x) = x^n$ with $n = 2, 3, 4, 5$ (satisfying conditions of theorem 3) are plotted in green, blue, red, brown respectively. Consider $(-\infty, 0]$ and $[0, \infty)$ separately.