Chapter 4. Mean Value Theorem and Applications

4.2 Monotonicity

When $f(x)$ is constant?

Theorem 1. *Let $f(x)$ be defined on (a, b). Then $f'(x) = 0$ identically on (a, b) (which means for all $x \in (a, b)$ iff $f(x) = \text{const}$ on (a, b).*

Proof. We know already (from definition) that $f(x) = \text{const}$ then $f'(x) = 0$.

Conversely, let $f'(x) = 0$ on (a, b). Consider $a < x < t < b$; according to MVT there exists $\xi \in (x, t)$ such that $\dfrac{f(t) - f(x)}{t - x} = f'(\xi)$; however $f'(\xi) = 0$ and therefore $f(t) = f(x)$ for all such x, t, which means exactly that $f(x) = \text{const}$ on (a, b). □

When $f(x)$ is monotone non-decreasing?

Definition 2. Let $f(x)$ be defined on (a, b). Then $f(x)$ is *monotone non-decreasing* on (a, b) iff $\forall x, t \in (a, b)\ x < t \implies f(x) \le f(t)$.
Further, $f(x)$ is *monotone non-increasing* on (a, b) iff $\forall x, t \in (a, b)\ x < t \implies f(x) \ge f(t)$.

Theorem 3. *Let $f(x)$ be defined and differentiable on (a, b). Then $f(x)$ is monotone non-decreasing on (a, b) iff $f'(x) \ge 0$ on (a, b) and $f(x)$ is monotone non-increasing on (a, b) iff $f'(x) \le 0$ on (a, b).*

Proof. Let $f(x)$ be monotone non-decreasing; then $\dfrac{f(t) - f(x)}{t - x} \ge 0$ as $x \ne t$ and therefore $f'(x) = \lim\limits_{t \to x} \dfrac{f(t) - f(x)}{t - x} \ge 0$.

Conversely, let $f'(x) = 0$ on (a, b). Consider $a < x < t < b$; according to MVT there exists $\xi \in (x, t)$ such that $\dfrac{f(t) - f(x)}{t - x} = f'(\xi)$; however $f'(\xi) \ge 0$ and therefore $f(t) \ge f(x)$ for all such x, t, which means exactly that $f(x)$ is monotone non-decreasing on (a, b).

The case of monotone non-increasing f is treated exactly like this. □

When $f(x)$ is monotone non-decreasing?

Definition 4. Let $f(x)$ be defined on (a, b). Then $f(x)$ is *monotone non-decreasing* on (a, b) iff $\forall x, t \in (a, b)\ x < t \implies f(x) \le f(t)$.
Further, $f(x)$ is *monotone non-increasing* on (a, b) iff $\forall x, t \in (a, b)\ x < t \implies f(x) \ge f(t)$.

Theorem 5. *Let $f(x)$ be defined and differentiable on (a, b). Then $f(x)$ is monotone non-decreasing on (a, b) iff $f'(x) \geq 0$ on (a, b) and $f(x)$ is monotone non-increasing on (a, b) iff $f'(x) \leq 0$ on (a, b).*

Proof. Let $f(x)$ be monotone non-decreasing; then $\dfrac{f(t) - f(x)}{t - x} \geq 0$ as $x \neq t$ and therefore $f'(x) = \lim\limits_{t \to x} \dfrac{f(t) - f(x)}{t - x} \geq 0$.

Conversely, let $f'(x) = 0$ on (a, b). Consider $a < x < t < b$; according to MVT there exists $\xi \in (x, t)$ such that $\dfrac{f(t) - f(x)}{t - x} = f'(\xi)$; however $f'(\xi) \geq 0$ and therefore $f(t) \geq f(x)$ for all such x, t, which means exactly that $f(x)$ is monotone non-decreasing on (a, b).

The case of monotone non-increasing f is treated exactly like this. □

When $f(x)$ is monotone increasing?

Definition 6. Let $f(x)$ be defined on (a, b). Then $f(x)$ is *monotone increasing* on (a, b) iff $\forall x, t \in (a, b)\ x < t \implies f(x) < f(t)$.
Further, $f(x)$ is *monotone decreasing* on (a, b) iff $\forall x, t \in (a, b)\ x < t \implies f(x) > f(t)$.

Often the world *strictly* is added to emphasize the strict inequality.

Theorem 7. *Let $f(x)$ be defined and differentiable on (a, b). Then $f(x)$ is monotone increasing on (a, b) provided $f'(x) > 0$ on (a, b) and $f(x)$ is monotone decreasing on (a, b) provided $f'(x) < 0$ on (a, b).*

Proof. Consider $a < x < t < b$; according to MVT there exists $\xi \in (x, t)$ such that $\dfrac{f(t) - f(x)}{t - x} = f'(\xi)$; however $f'(\xi) > 0$ and therefore $f(t) > f(x)$ for all such x, t, which means exactly that $f(x)$ is monotone increasing on (a, b).

The case of monotone decreasing f is treated exactly like this. □

Remark 8. The first part of the proof of theorem 5 does not work in full here: even if $\dfrac{f(t) - f(x)}{t - x} > 0$ as $x \neq t$, we can conclude that the limit of it as $t \to x$ is non-negative but we cannot conclude that it is positive. And in fact, while $f(x) = x^3$ is monotone increasing function, its derivative $f'(x) = 3x^2$ is 0 as $x = 0$.

So, we have a gap between sufficient condition ($f'(x) > 0$) and necessary condition $f'(x) \geq 0$ of the strict monotonicity.

Theorem 9. *Let $f(x)$ be defined and differentiable on (a, b). Then $f(x)$ is monotone increasing on (a, b) iff $f'(x) \geq 0$ on (a, b) and each interval $(\alpha, \beta) \subset (a, b)$ contains x s.t. $f'(x) > 0$.*
Further $f(x)$ is monotone decreasing on (a, b) iff $f'(x) \leq 0$ on (a, b) and each interval $(\alpha, \beta) \subset (a, b)$ contains x s.t. $f'(x) < 0$.

Proof. Let $f(x)$ be monotone increasing but there is an interval (α, β) on which $f'(x) = 0$ identically. But then $f(x) = \text{const}$ on (α, β) and cannot be monotone increasing.

Conversely, as $f'(x) \geq 0$ on (a, b) function $f(x)$ is monotone non-decreasing. Assume that it is not monotone increasing, then exist α, β s.t. $\alpha < \beta$ but $f(\alpha) \geq f(\beta)$. Then since $f(\alpha) \leq f(x) \leq f(\beta)$ on (α, β) we conclude that $f(x) = f(\alpha) = f(\beta)$ on (α, β) and therefore $f'(x) = 0$ there. $\square$

Example 1. Find intervals of monotonicity of the following functions:

$(a)\ x^3 - 3x$ $\quad (b)\ x^3 + 3x$ $\quad (c)\ \dfrac{1}{x}$ $\quad (d)\ \dfrac{1}{x^2}$

$(e)\ x + \dfrac{1}{x}$ $\quad (f)\ x - \dfrac{1}{x}$ $\quad (g)\ \sin\left(\dfrac{1}{x}\right)$ $\quad (h)\ \tan x$