Chapter 4. Mean Value Theorem and Applications

4.1 MVT

Maxima and Minima (first look)

Theorem 1. *(i) Let $f(x)$ be right-differentiable in c and $f'(c) > 0$. Then there exists $\delta > 0$ such that $f(x) > f(c)$ as $c < x < c + \delta$.*

(ii) Let $f(x)$ be right-differentiable in c and $f'(c) < 0$. Then there exists $\delta > 0$ such that $f(x) < f(c)$ as $c < x < c + \delta$.

(iii) Let $f(x)$ be left-differentiable in c and $f'(c) > 0$. Then there exists $\delta > 0$ such that $f(x) < f(c)$ as $c - \delta < x < c$.

(iv) Let $f(x)$ be left-differentiable in c and $f'(c) < 0$. Then there exists $\delta > 0$ such that $f(x) > f(c)$ as $c - \delta < x < c$.

Proof. Let us prove (i) (other statements are proven similarly). Since $\lim\limits_{x\to c^+} \dfrac{f(x) - f(c)}{x - c} > 0$ then exists $\delta > 0$ s.t. $\dfrac{f(x) - f(c)}{x - c} > 0$ as $c < x < c + \delta$ and then $f(x) - f(c) > 0$ there. □

Definition 2. Let $f(x)$ be defined on $[a, b]$. Then

(i) $c \in [a, b]$ is a (point of) maximum of $f(x)$ if $f(c) \geq f(x)$ for all $x \in [a, b]$.

(ii) $c \in [a, b]$ is a (point of) minimum of $f(x)$ if $f(c) \leq f(x)$ for all $x \in [a, b]$.

Both maximumum and minimum are called *extremums.* We consider also *local* extremums (below). In contrast to them what is defined in Definition 2 are called *global* or *absolute* extremums.

Definition 3. Let $f(x)$ be defined on $[a, b]$. Then

(i) $c \in [a, b]$ is a (point of) a local maximum of $f(x)$ if there exists $\delta > 0$ s.t. $f(c) \geq f(x)$ for all $x \in [a, b] \cap (c - \delta, c + \delta)$.

(ii) $c \in [a, b]$ is a (point of) a local minimum of $f(x)$ if there exists $\delta > 0$ s.t. $f(c) \leq f(x)$ for all $x \in [a, b] \cap (c - \delta, c + \delta)$

Here one should check intervals: $(c - \delta, c + \delta)$ as $c \in (a, b)$, $[a, a + \delta)$ as $c = a$, $(b - \delta, b]$ as $c = b$.

Theorem 4. *Let $f(x)$ be defined on $[a, b]$ and differentiable at c. Then*

(i) If $c \in (a, b)$ is a (point of) a local maximum/minimum of $f(x)$ then $f'(c) = 0$.

(ii) If $c = a$ is a (point of) a local maximum (minimum) of $f(x)$ then $f'(c) \leq 0$ ($f'(c) \geq 0$ respectively).

(iii) If $c = b$ is a (point of) a local maximum (minimum) of $f(x)$ then $f'(c) \geq 0$ ($f'(c) \leq 0$ respectively).

Proof. If $c = a$ and $f'(c) > 0$ then by Theorem 1 $f(x) > f(c)$ as $c < x < c + \delta$ and c cannot be a point of a local maximum. Similar arguments prove the rest of (ii) and (iii).
Therefore, if $c \in (a, b)$ is a point of a local minimum or maximum, we must have $f'(c) \geq 0$ and $f'(c) \leq 0$ simultaneously and thus $f'(c) = 0$. □

Definition 5. Point c s.t. $f'(c) = 0$ are called *critical* or *stationary* points of $f(x)$.

Therefore

Corollary 6. *To find points of local minima and maxima of $f(x)$ on $[a, b]$ one should check*

1. *Stationary points of $f(x)$;*
2. *Points where $f'(x)$ does not exist;*
3. *Endpoints (a and b).*

I remind that continuous on the closed interval function $f(x)$ must have minima and maxima on the finite closed interval $[a, b]$.

Example 1. Check functions

$$f(x) = \frac{x}{x^2+1}; \qquad \text{on } (-\infty, \infty); \qquad f(x) = \frac{x^2}{(x^2+1)^2} \qquad \text{on } (-\infty, \infty);$$
$$f(x) = |x| \qquad \text{on } (-1, 1); \qquad f(x) = \frac{x}{2} + |x| \qquad \text{on } (-1, 1)$$
$$f(x) = 2x^{\frac{2}{3}} - 3x \qquad \text{on } (-\frac{3}{2}, \frac{3}{2}); \qquad f(x) = \sin^3 x - \frac{3}{4}\sin x \qquad \text{on } (-\infty, \infty);$$

MVT

We start from MVT light:

Theorem 7. *(Rolle) Let $f(x)$ be defined and continuous on $[a, b]$ and differentiable on (a, b). Assume that $f(a) = f(b)$. Then there exists $c \in (a, b)$ s.t. $f'(c) = 0$*

Proof. Since f is continuous on $[a, b]$ it reaches maximum at some point. Either this is $c \in (a, b)$ and then $f'(c) = 0$ by virtue of theorem 4 (and theorem 7 is proven) or maximum is reached in one (and therefore both) of points a, b.
Also f reaches minimum at some point. Either this is $c \in (a, b)$ and then $f'(c) = 0$ by virtue of theorem 4 (and theorem 7 is proven) or minimum is reached in one (and therefore both) of points a, b.
So, theorem 7 is proven unless $\max_{[a,b]} f = \min_{[a,b]} f = f(a) = f(b)$ in which case $f = \text{const}$ on $[a, b]$ and $f'(c) = 0$ at every point $c \in (a, b)$.

□

Theorem 8. *(MVT, Lagrange) Let $f(x)$ be defined and continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ s.t.*

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \tag{1}$$

Proof. Note that $k = \frac{f(b)-f(a)}{b-a}$ is the slope of the straight line connecting $(a, f(a))$ and $(b, f(b))$ on the graph of f. Also note that $g(x) - kx$ satisfies conditions of Rolle' theorem: $g(b) - g(a) = f(b) - kb - f(a) + ka = f(b) - f(a) - k(b - a) = 0$.
Then there exists $c \in (a, b)$ with $g'(c) = 0$. But $g'(c) = f'(c) - kc$ and (1) is equivalent to $g'(c) = 0$.

□

Corollary 9. *Let $f(x)$ be defined and continuous on $[a, b]$ and differentiable on (a, b). Assume that $f(x_1) = f(x_2) = \cdots = f(x_N) = 0$ with $a \le x_1 < x_2 < \cdots < x_N \le b$.*
Then there exist $y_1, y_2, \dots, y_{N-1}$ with $x_1 < y_1 < x_2 < \dots y_{N-1} < \dots x_{N-1} << x_N$ with $f'(y_1) = \cdots = f'(y_{N-1}) = 0$.

Corollary 10. *Polynomial $f(x)$ of degree M has no more than N distinct real roots.*

Proof. If f has $N + 1$ or more distinct real roots then $f'(x)$ would have at least N distinct real roots, then $f''(x)$ would have at least N distinct real roots etc, with $f^{(n-1)}(x)$ having two real roots which is not possible as it is a linear function.

□

Example 2. Can we apply Rolle's theorem to $f(x) = |x|$ on

$(a)\ f(x) = \lvert x \rvert;$	on $(-1, 1);$	$(b)\ f(x) = x^2 - 2x$	on $(0, 2);$
$(c)\ f(x) = \sin\frac{1}{x}$	on $(0, 1);$	$(d)\ f(x) = x \cdot \sin\frac{1}{x}$	on $(0, 1);$

If yes, what is conclusion in each case?

(a) No, $|x|$ is not differentiable at 0;
(b) Yes, and one can check that $f'(1) = 0$;
(c,d) Yes, for example, since $f(\frac{1}{n\pi}) = 0$ for all $n \in \mathbb{Z}$ there would be $y_n \in (\frac{1}{n\pi}, \frac{1}{(n-1)\pi})$ with $f'(y_n) = 0$; one can check that $y_n = \frac{2}{(2n-1)\pi}$ in (c) but in (d) we get equation $z = \cot z$ for $z = 1/y$ and this equation has an infinite series of solutions.
Section 4.2 will be covered in the next handout