Chapter 3. Differentiation

3.5 Chain rule

Main Theorem

Theorem 1. *Let g be differentiable at x and f be differentiable at $y = g(x)$. Then $f \circ g$ is differentiable at x and*

$$(f \circ g)'(x) = f'\bigl(g(x)\bigr) \cdot g'(x). \tag{1}$$

Proof. Proof at textbook (Supplement to section 3.5, pp 167–168.) □

Remark 2. (i) One can rewrite the right-hand expression in (1) as $(f' \circ g)(x) \cdot g'(x)$;
(ii) Denoting $g(x) = y$ and $f(y) = z$ one can rewrite (1) as

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} \tag{2}$$

which vindicates notation $y' = \frac{dy}{dx}$. While dx, dy, dz are not defined so far, derivative could be treated formally as the ratio of dy and dx.
(iii) However it does not vindicate notations $y'' = \frac{d^2y}{dx^2}$, $y''' = \frac{d^3y}{dx^3}$.

Main Example

Example 1.

$$\Bigl(\sin^4\bigl(1 + (3 + 7x^2)^{-2}\bigr)^3\Bigr)' = \Bigl(\Bigl(\sin\bigl(1 + (3 + 7x^2)^{-2}\bigr)^3\Bigr)^4\Bigr)' =$$

because external function is ^{4} - avoid possible confusion

$$=4\Bigl(\sin\bigl(1 + (3 + 7x^2)^{-2}\bigr)^3\Bigr)^3 \cdot \Bigl(\sin\bigl(1 + (3 + 7x^2)^{-2}\bigr)^3\Bigr)' =$$
$$=4\sin^3\bigl(1 + (3 + 7x^2)^{-2}\bigr)^3 \cdot \cos\bigl(1 + (3 + 7x^2)^{-2}\bigr)^3 \cdot \Bigl(\bigl(1 + (3 + 7x^2)^{-2}\bigr)^3\Bigr)' =$$
$$=4\sin^3\bigl(1 + (3 + 7x^2)^{-2}\bigr)^3 \cdot \cos\bigl(1 + (3 + 7x^2)^{-2}\bigr)^3 \cdot 3\bigl(1 + (3 + 7x^2)^{-2}\bigr)^2 \cdot \bigl(1 + (3 + 7x^2)^{-2}\bigr)' =$$
$$= -24\sin^3\bigl(1 + (3 + 7x^2)^{-2}\bigr)^3 \cdot \cos\bigl(1 + (3 + 7x^2)^{-2}\bigr)^3 \cdot \bigl(1 + (3 + 7x^2)^{-2}\bigr)^2 \cdot (3 + 7x^2)^{-3} \cdot (3 + 7x^2)' =$$
$$= -336\sin^3\bigl(1 + (3 + 7x^2)^{-2}\bigr)^3 \cdot \cos\bigl(1 + (3 + 7x^2)^{-2}\bigr)^3 \cdot \bigl(1 + (3 + 7x^2)^{-2}\bigr)^2 \cdot (3 + 7x^2)^{-3}x$$