

Chapter 10. Limits and Continuity

10.1 Sharp bounds

Definitions

Consider set $S \subset \mathbb{R}$ bounded from above:

$$\forall x \in S \quad x \leq M$$

and I remind that M is an upper bound of S .

Clearly if M is an upper bound of S and $M' > M$ then M' is also an upper bound. One can wonder if the least upper bound exists.

Not always the case if we consider the set of rational numbers \mathbb{Q} : F. e. there is no the least upper bound $r \in \mathbb{Q}$ for the set of numbers such that $x^2 < 2$.

This is exactly why we need real numbers:

Axiom 1. (Completeness) Each bounded from above set $S \subset \mathbb{R}$ has the least upper bound $M \in \mathbb{R}$: if M' is another upper bound for S then $M' \geq M$.

I remind that

Axiom 2.

- (**Order**) Relations $>$, \geq , $<$, \leq are defined for real numbers and have usual properties;
- (**Density of rational numbers**) If $r < r'$ are two real numbers then there exists rational number r'' between them: $r < r'' < r'$.

To be really tough one should construct \mathbb{R} based on \mathbb{Q} .

Definition 1.

- The least upper bound of set S is denoted $\text{lub } S$ (in the textbook) or $\text{sup } S$ (standard) and called supremum;
- If S is not bounded from above then $\text{sup } S = +\infty$;
- If $S = \emptyset$ then $\text{sup } S = -\infty$;
- If $\text{sup } S \in S$ it could be denoted by $\text{max } S$ and called maximum.

Also consider set $S \subset \mathbb{R}$ bounded from below:

$$\forall x \in S \quad x \geq m$$

and I remind that m is a lower bound of S .

Clearly if m is a lower bound of S and $m' < m$ then m' is also a lower bound.

Theorem 2. Each bounded from below set $S \subset \mathbb{R}$ has the greatest lower bound $m \in \mathbb{R}$: if m' is another lower bound for S then $m' \leq m$.

Proof. Reflecting \mathbb{R} at 0 and applying Completeness Axiom. □

Definition 3.

- The greatest lower bound of set S is denoted $\text{glb } S$ (in the textbook) or $\text{inf } S$ (standard) and called infimum;
- If S is not bounded from below then $\text{inf } S = -\infty$;
- If $S = \emptyset$ then $\text{inf } S = +\infty$;
- If $\text{inf } S \in S$ it could be denoted by $\text{min } S$ and called minimum.

Theorems

Theorem 4. Let $S \neq \emptyset$. Then

- If $M = \text{sup } S < \infty$ then

$$\forall \epsilon > 0 \quad \exists x \in S : \quad M - \epsilon < x \leq M;$$

- If $m = \text{inf } S > -\infty$ then

$$\forall \epsilon > 0 \quad \exists x \in S : \quad m \leq x < m + \epsilon;$$

Proof. Otherwise $M - \epsilon$ or $m + \epsilon$ would be upper or lower bounds respectively. □

Theorem 5. Let $S \neq \emptyset$. Then

$\text{inf } S \leq \text{sup } S$ and

$\text{inf } S = \text{sup } S$ iff S consists of exactly one element.

Proof. If $S \neq \emptyset$ then exists $x \in S$ and therefore $\text{inf } S \leq x \leq \text{sup } S$.

If $S = \{x\}$ then $\text{inf } S = x = \text{sup } S$. If S contains at least two elements, say x and y , $x < y$ then $\text{inf } S \leq x < y \leq \text{sup } S$. □

Continuity: Hard theorems

Uniform continuity

Definition 6. Function $f(x)$ defined on $[a, b]$ is uniformly continuous on $[a, b]$ iff

$$\forall \epsilon > 0 \quad \exists \delta > 0 : \quad \forall x, y \in [a, b] \quad |x - y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

For uniform continuous function δ depends only on ϵ but not on y : compare with

$$\forall y \in [a, b] \forall \epsilon > 0 \exists \delta > 0 : \forall x \in [a, b] |x - y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

Theorem 7. *Function $f(x)$ continuous on closed finite interval $[a, b]$ is uniformly continuous on $[a, b]$.*

Proof. By contradiction. Assume that this is not the case. Then

$$\exists \epsilon > 0 : \forall \delta > 0 \exists x, y \in [a, b] : |x - y| < \delta \ \& \ |f(x) - f(y)| \geq \epsilon.$$

Pick this “bad” $\epsilon > 0$ and let

$$(1) \quad S = \{c : \forall \delta > 0 \exists x, y \in [a, c] : |x - y| < \delta \ \& \ |f(x) - f(y)| \geq \epsilon\}.$$

Let $d = \inf S$; then $d \in [a, b]$. Clearly: in the “rogue pair” (x, y) satisfying (1), at least one element could be chosen arbitrarily close to d : it must be neither necessarily greater than $d + \delta_0$ with fixed $\delta_0 > 0$ (then $\inf S \geq d + \delta_0$) nor less than $d - \delta_0$ with fixed $\delta_0 > 0$ (then $\inf S \leq d - \delta_0$). So, o

$$(2) \quad \forall \delta > 0 \exists x, y \in [a, c] : |x - d| \leq \delta \ \& \ |x - y| < \delta \ \& \ |f(x) - f(y)| \geq \epsilon.$$

Then in (2) $|x - d| < 2\delta$, $|y - d| < 2\delta$. However this is impossible: due to continuity of $f(x)$ in d there exists $\delta > 0$ such that $|x - d| < 2\delta \implies |f(x) - f(d)| < \frac{1}{3}\epsilon$. \square

Example 1. Functions x^2 on $[0, \infty)$ or $\frac{1}{x}$ on $(0, 1]$ are continuous but not uniformly continuous on these intervals. Finite and closed requirement cannot be relaxed.

Proposition 8. *Function $f(x)$ uniformly continuous on any finite interval is bounded here.*

Proof. Take $\epsilon = 1$. Then $|f(x) - f(y)| < 1$ as $|f(x) - f(y)| < \delta$ and since one can go through $[a, b]$ by $\lceil \frac{b-a}{\delta} \rceil + 1$ lesser than δ $|f(x) \leq |f(a)| + \lceil \frac{b-a}{\delta} \rceil + 1$. \square

Maximum and minimum

Definition 9. Function $f(x)$ has maximum at c iff $f(x) \leq f(c) \ \forall x \in [a, b]$.
Then

$$f(c) = \max_{x \in [a, b]} f(x).$$

Function $f(x)$ has strict maximum at c iff $f(x) < f(c) \ \forall x \in [a, b], x \neq c$.

Definition 10. Function $f(x)$ has minimum at c iff $f(x) \geq f(c) \ \forall x \in [a, b]$.
Then

$$f(c) = \min_{x \in [a, b]} f(x).$$

Function $f(x)$ has strict minimum at c iff $f(x) > f(c) \ \forall x \in [a, b], x \neq c$.

Theorem 11. *Let function $f(x)$ be continuous on closed finite interval $[a, b]$. Then there exist $c, d \in [a, b]$ which are points of maximum and minimum of $f(x)$ on $[a, b]$ respectively.*

Proof. Easy to see that $c = \sup S$ with $S = \{y \in [a, b] : \sup_{[a, y]} f(x) < M\}$ where $M = \sup_{x \in [a, b]} f(x)$ is a finite number due to theorem 7 and proposition 8.

Similarly, $d = \sup R$ with $R = \{y \in [a, b] : \inf_{[a, y]} f(x) > m\}$ where $m = \inf_{x \in [a, b]} f(x)$.

Then for arbitrarily small $\epsilon > 0$ let $\delta > 0$ be such that $|x - c| < \delta \implies |f(x) - f(c)| < \epsilon$.

Assume first that $a < c < b$ and select $\delta < \min(b - c, c - a)$. Since $\sup_{[a, c+\delta]} f(x) = M$ and $\sup_{[a, c-\delta]} f(x) < M$ we conclude that $\sup_{[c-\delta, c+\delta]} f(x) = M$ and therefore there exists $x \in (c - \delta, c + \delta)$ such that $f(x) > M - \epsilon$. Since $|f(x) - f(c)| < \epsilon$ we conclude $f(c) > M - 2\epsilon$. This holds for arbitrarily small $\epsilon > 0$ and therefore $f(c) \geq M$. Since $f(c) \leq M$ we conclude that $f(c) = M$.

As $c = b$ the same arguments work with $c + \delta$ replaced by b . As $c = a$ we the same arguments work with $c - \delta$ replaced by a .

Since $\min f(x) = -\max(-f(x))$, $\inf f(x) = -\sup(-f(x))$ we conclude that also $f(d) = m$. There could be other points of maximum and minimum but c, d are the left-most of them. \square

Example 2. Functions x^2 and $-\frac{1}{x^2 + 1}$ on $[0, \infty)$ or $\frac{1}{x}$ on $(0, 1]$ are continuous but they do not reach maximum there. Finite and closed requirement cannot be relaxed.

Intermediate value

Theorem 12. *Let function $f(x)$ be continuous on closed finite interval $[a, b]$ and $f(a) < L < f(b)$ (or $f(a) < L < f(b)$). Then there exists $c \in (a, b)$ such that $f(c) = L$.*

Proof. Assume that $f(a) < f(b)$. Clearly $f(c) = L$ as $c = \{y : f(y) < L\}$. There could be other points x with $f(x) = L$ but c is the left-most of them. \square

Corollaries

Corollary 13.

- (i) *Function $f(x)$, continuous on closed finite interval $[a, b]$, is bounded there.*
- (ii) *Range of $f(x)$ defined and continuous on closed finite interval $[a, b]$ is a finite closed interval $[m, M]$, with $m = \min_{x \in [a, b]} f(x)$ and $M = \max_{x \in [a, b]} f(x)$.*