Chapter 2. Limits and Continuity

2.4 Continuity (end)

One-sided continuity

Definition 5. Function $f(x)$ defined on $(a,c]$ is *continuous at point c from the left* iff

$$\forall \epsilon > 0 \; \exists \delta > 0 : \; c - \delta < x \leq c \implies |f(x) - f(c)| < \epsilon.$$

Function $f(x)$ defined on $[c,b)$ is *continuous at point c from the right* iff

$$\forall \epsilon > 0 \; \exists \delta > 0 : \; c - \delta < x \leq c \implies |f(x) - f(c)| < \epsilon.$$

Theorems

Try to prove and understand three following theorems:

Theorem 6. *Function $f(x)$ continuous at point c from the left/right*

$$\lim_{x \to c^{\mp}} f(x) = f(c).$$

Theorem 7. *Function $f(x)$ continuous at point c iff it is continues at point c both from the left and from the right.*

Theorem 8. *Let $f(x), g(x)$ be continuous at point c from the left/right. Then*
(i) $(f \pm g)(x)$, $\alpha f(x)$ are continuous at point c from the left/right ($\alpha = \text{const}$);
(ii) $(f)(x)$ is continuous at point c from the left/right;
(iii) If $g(c) \neq 0$ then $\frac{f(x)}{g(x)}$ is continuous at point c from the left/right.

Theorem 9. *Let $g(x)$ be continuous at point c from the left/right and $f(y)$ be continuous[a] at point $k = g(c)$. Then $(f \circ g)(x) = f\big(g(x)\big)$ is continuous at point c from the left/right.*

[a] two sided continuity is needed here. Explain why.

Continuity on intervals

Definition 10. Let $a < b$. Then

- Function $f(x)$ is *continuous on interval* (a, b) iff it is continuous at each point of (a, b);
- Function $f(x)$ is *continuous on segment* $[a, b]$ iff it is continuous at each point of (a, b) and also it is continuous from the left at b and from the right at a;
- Function $f(x)$ is *continuous on* $(a, b]$ iff it is continuous at each point of (a, b) and also it is continuous from the left at b;

  Function $f(x)$ is *continuous on* $[a, b)$ iff it is continuous at each point of (a, b) and also it is continuous from the right at a.

Theorem 11. *Function continuous on some interval I is continuous on every smaller interval I'. Function continuous on intervals I' and I'' having at least one common point is continuous on interval $I' \cup I''$.*

More serious theorems will be formulated in the next class. Notion of continuity on interval is actually much more important than the one of continuity at one point.

Examples

Example 5. Define values of the following functions at 0 so they will be continues there from the left/right (or indicate that this is impossible):

$$f(x) = 1 + \frac{x}{|x|}; \qquad g(x) = \frac{1}{x} f(x); \qquad h(x) = \frac{1}{x|x|}.$$