# Chapter 2. Limits and Continuity

## 2.4 Continuity

### Definition

Try to prove and understand three following theorems:

**Definition 1.** Function $f(x)$ defined on $(a, b) \ni c$ is *continuous at point c* iff

$$\forall \epsilon > 0\ \exists \delta > 0 :\ |x - c| < \delta \implies |f(x) - f(c)| < \epsilon.$$

### Theorems

Try to prove and understand three following theorems:

**Theorem 2.** *Function $f(x)$ defined on $(a, b) \ni c$ is continuous at point $c$ iff*

$$\lim_{x \to c} f(x) = f(c).$$

*Proof.* Proof follows immediately from the definitions of limit and continuity; important is that inequality $|f(x) - f(c)| < \epsilon$ for $x = c$ holds for sure. □

**Theorem 3.** *Let $f(x), g(x)$ be continuous at point $c$. Then*
*(i) $(f \pm g)(x)$, $\alpha f(x)$ are continuous at point $c$ ($\alpha = \text{const}$);*
*(ii) $(f)(x)$ is continuous at point $c$;*
*(iii) If $g(c) \neq 0$ then $\frac{f(x)}{g(x)}$ is continuous at point $c$.*

*Proof.* Proof follows immediately from the properties of limit and theorem 2. □

**Theorem 4.** *Let $g(x)$ be continuous at point $c$ and $f(y)$ be continuous at point $k = g(c)$. Then $(f \circ g)(x) = f\big(g(x)\big)$ is continuous at point $c$.*

*Proof.* Since $f(x)$ is continuous at point $k = g(c)$

$$\forall \epsilon > 0\ \exists \eta = \eta(\epsilon) > 0 :\ |y - k| < \eta \implies |f(y) - f(k))| < \epsilon$$

and since $g(x)$ is continuous at point $c$

$$\forall \eta > 0\ \exists \delta = \delta(\eta) > 0 :\ |x - c| < \delta \implies |g(x) - g(c)| < \eta.$$

Then

$$\forall \epsilon > 0\ \exists \delta = \delta\big(\eta(\epsilon)\big) > 0 :\ |x - c| < \delta \implies |f\big(g(x)\big) - f\big(g(c)\big)| < \epsilon.$$

□

## Examples

**Example 1.** (i) Polynomial $P(x)$ is continuous everywhere;
(ii) Rational function $\frac{P(x)}{Q(x)}$ is continuoyus where $Q(x)\neq 0$.

**Example 2.** (i) Function $f(x)=\frac{\sin x}{x}$ is defined as $x\neq 0$ and continuous there. However we know that $\lim_{x\to 0}\frac{\sin x}{x}=1$ and therefore [re]defining $f(0)=1$:

$$f(x)=\begin{cases}\dfrac{\sin x}{x} & x\neq 0;\\ 1 & x=0\end{cases}$$

we get continuous function. We call discontinuity of $\frac{\sin x}{x}$ at 0 *remoivable.*

**Example 3.** Find where the following functions are discontinuous and which discontinuities are removable:

$$\begin{array}{lll}\tan x, & \sec x, & \\ \cot x, & \csc x, & \\ \sin\left(\dfrac{1}{x}\right), & x\sin\left(\dfrac{1}{x}\right), & \\ \dfrac{1}{|x|}, & \dfrac{x}{|x|}, & \dfrac{x^3}{|x|}.\end{array}$$

**Example 4.** Let

$$f(x)=\begin{cases}1 & x\text{ is rational}\\ 0 & x\text{ is irrational}\end{cases}$$

Where $f(x)$ is continuous? Where $xf(x)$ is continuous? Where $f(x)\sin x$ is continuous?