Chapter 2. Limits and Continuity

2.3 Main theorems about limits.

Theorems. I.

Try to prove and understand three following theorems:

Theorem 1. *(Uniqueness) If* $\lim_{x\to c} f(x) = L$ *and* $\lim_{x\to c} f(x) = K$ *then* $K = L$.

This theorem remains true for one-sided limits as well. This remark applies to almost all theorems (with exception of 3). But some of them need minor modifications. Which?

Next theorem links limit and boundedness.

Theorem 2. *(i) If* $\lim_{x\to c} f(x)$ *exists then* f *is bounded on* $(c-\epsilon, c)\cup(c, c+\epsilon)$ *for some* $\epsilon > 0$.

(ii) If $\lim_{x\to c^-} f(x)$ *exists then* f *is bounded on* $(c-\epsilon, c)$ *for some* $\epsilon > 0$.

(iii) If $\lim_{x\to c^+} f(x)$ *exists then* f *is bounded on* $(c, c+\epsilon)$ *for some* $\epsilon > 0$.

Next theorem links two-sided and one-sided limits.

Theorem 3. *(i)* $\lim_{x\to c} f(x) = L$ *iff* $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} = L$.

(ii) In particular $\lim_{x\to c} f(x)$ *exists iff exist both limits* $\lim_{x\to c^-} f(x)$ *and* $\lim_{x\to c^+} f(x)$ *and they are equal.*

Theorem 4. $\lim_{x\to c} f(x) = L$ *iff* $\lim_{x\to c}\big(f(x) - L\big) = 0$.

Theorems. II.

Theorem 5. *If* $\lim_{x\to c} f(x) = 0$ *and* $g(x)$ *is bounded on* $(a, b) \ni c$ *then* $\lim_{x\to c}\big(f(x)g(x)\big) = 0$.

Proof. Since $g(x)$ is bounded on $(a, b) \ni c$ then $|g(x)| \leq M$ on $(c - \delta_1, c + \delta_1)$ for some $\delta_1 > 0$. Since $\lim_{x\to c} f(x) = 0$ for any $\epsilon > 0$ there exists $\delta_2 > 0$ such that $|f(x)| < \frac{\epsilon}{M}$ as $0 < |x - c| < \delta_2$. Really, since $\epsilon/(M + 1) > 0$ we can plug it into definition of the limit instead of ϵ. *Instead of* ϵ *we can plug any number* > 0.
Let us take $\delta = \min(\delta_1, \delta_2)$. Then $|f(x)g(x)| < \dfrac{\epsilon}{M+1} \cdot M = \epsilon$ as $0 < |x - c| < \delta$. □

Theorem 6. *If* $\lim_{x\to c} f(x) = L$ *and* $\lim_{x\to c} g(x) = M$ *then*

(i) $\lim_{x\to c}\big(f(x) \pm g(x)\big) = L \pm M$;

(ii) $\lim_{x\to c}\big(kf(x)\big) = kL$ *as* $k = \text{const}$;

(iii) $\lim_{x\to c}\big(f(x)g(x)\big) = LM$;

Proof. (i) We can plug $\frac{\epsilon}{2}$ instead of $\epsilon > 0$ in the definition of the limit. Then there exist $\delta_1 > 0$ "serving" $f(x)$ and $\delta_2 > 0$ "serving" $g(x)$ (both for $\frac{\epsilon}{2}$). But then

$$|\bigl((x) \pm g(x)\bigr) - (L \pm M)| \le |f(x) - L| + |g(x) - M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

(ii) $\lim_{x\to c} f(x) = L \implies \lim_{x\to c}\bigl(f(x) - L\bigr) = 0 \implies \lim_{x\to c} k\bigl(f(x) - L\bigr) = 0$ (because constant is a bounded function) or, equivalently $\lim_{x\to c}\bigl(kf(x) - kL\bigr)$ which implies (ii).

(ii) Note that

$$f(x)g(x) - LM = \bigl(f(x) - L\bigr)g(x) + L\bigl(g(x) - M\bigr).$$

Here $\lim_{x\to c}\bigl(f(x)-L\bigr) = 0$ and $g(x)$ is bounded on $(c-\delta, c+\delta)$; then $\lim_{x\to c}\bigl(f(x)-L\bigr)g(x) = 0$. Similarly, $\lim_{x\to c} L\bigl(g(x)-M\bigr) = 0$. This implies that $\lim_{x\to c}\bigl(f(x)g(x)-LM\bigr) = 0$ which implies (iii).

□

Theorem 7. *If* $\lim_{x\to c} f(x) = L$ *and* $\lim_{x\to c} g(x) = M \ne 0$ *then* $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$.

Proof. First, $g(x)$ is *disjoint* from 0 as $0 < |x - c| < \delta$ for some $\delta > 0$; this means that $|g(x)| \ge \epsilon_1$ for some $\epsilon_1 > 0$. Really, taking $\epsilon = |M|/2$ we conclude that $|g(x) - M| < |M|/2$ and therefore $|g(x)| \ge M/2$ as $0 < |x - c| < \delta$; therefore $\frac{1}{g(x)}$ is bounded there.
Then

$$\frac{1}{g(x)} - \frac{1}{M} = -\bigl(g(x) - M\bigr) \cdot \frac{1}{g(x)} \cdot \frac{1}{M}$$

where $\lim_{x\to c}\bigl(g(x) - M\bigr) = 0$ and $\frac{1}{g(x)}$ is a bounded function. Then $\lim_{x\to c}\Bigl(\frac{1}{g(x)} - \frac{1}{M}\Bigr) = 0$ which implies theorem in the special case $f(x) = 1$ i.e. for $1/g(x)$. However, now in the general case we can apply theorem 6(iii).

□