

Chapter 2. Limits and Continuity

2.2 Definition of limit

Bounded functions

We will need the notion now but explore in details later

Function f defined on $S \subset \mathbb{R}$ is *bounded* on S iff

$$\exists M : \forall x \in S \quad |f(x)| \leq M.$$

Number M above is *bound* of f on S .

What does it mean in plain English? Negate this definition:

Function f defined on $S \subset \mathbb{R}$ is *unbounded* on S iff

$$\forall M \exists x \in S \quad |f(x)| > M.$$

What does it mean in plain English?

Which functions below are bounded on indicated domains:

$$f(x) = \sin\left(\frac{1}{x}\right) \quad \text{on } (0, \infty);$$

$$f(x) = x \sin\left(\frac{1}{x}\right) \quad \text{on } (0, \infty);$$

$$f(x) = \frac{1}{\sin(x^2)} \quad \text{on } (1, \sqrt{\pi});$$

Definition of limit

Consider function f defined on $S = (a, b) \setminus \{c\} = (a, c) \cup (c, b)$ where $a < c < b$. Then

$$\lim_{x \rightarrow c} f(x) = L$$

(or $f(x) \rightarrow L$ as $x \rightarrow c$) iff

$$\forall \epsilon > 0 \exists \delta > 0 \forall x \quad 0 < |x - c| < \delta \implies |f(x) - L| < \epsilon.$$

What does it mean in plain English? Negate this definition:

Consider function f defined on $S = (a, b) \setminus \{c\} = (a, c) \cup (c, b)$ where $a < c < b$. Then L is not a limit of $f(x)$ as $x \rightarrow c$ (or $f(x) \not\rightarrow L$ as $x \rightarrow c$) iff

$$\exists \epsilon > 0 \quad \forall \delta > 0 \quad \exists x \quad 0 < |x - c| < \delta \quad \& \quad |f(x) - L| \geq \epsilon.$$

What does it mean in plain English? Consider examples from the textbook. Note that the definition of limit above does not cover infinite limits. Later we extend this definition to cover infinite limits but we will need to modify some theorems.

Left-hand limits

Consider function f defined on $S = (a, c)$ where $a < c$. Then

$$\lim_{x \rightarrow c^-} f(x) = L$$

(or $f(x) \rightarrow L$ as $x \rightarrow c^-$) iff

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall x \quad (c - x) < \delta \implies |f(x) - L| < \epsilon.$$

What does it mean in plain English? Negate this definition, Consider examples from the textbook.

Right-hand limits

Consider function f defined on $S = (c, b)$ where $c < b$. Then

$$\lim_{x \rightarrow c^+} f(x) = L$$

(or $f(x) \rightarrow L$ as $x \rightarrow c^+$) iff

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall x \quad (x - c) < \delta \implies |f(x) - L| < \epsilon.$$

What does it mean in plain English? Negate this definition, Consider examples from the textbook.