APM 346Y, Mid-Term Test. Nov 08, 00

Problems (few misprints fixed).

MT-1 By method of characteristics solve either (a) (*4 pts*) or (b) (*7 pts*)

(a) $$\begin{cases} u_{tt} - 9u_{xx} = 0, & x > 0, t > 0, \\ u|_{x=0} = 0, & \\ u|_{t=0} = \cos x, & u_t|_{t=0} = \sin x \end{cases};$$

(b) $$\begin{cases} u^-_{tt} - u^-_{xx} = 0, & x < 0, t > 0, \\ u^+_{tt} - 9u^+_{xx} = 0, & x > 0, t > 0 \\ u^-|_{x=0} = u^+|_{x=0}, & u^-_x|_{x=0} = u^+_x|_{x=0} - \cos t, \\ u^-|_{t=0} = u^-_t|_{t=0} = 0 & x < 0, \\ u^+|_{t=0} = u^+_t|_{t=0} = 0 & x > 0. \end{cases}$$

Hint What regions, where different formulae hold, are here?

MT-2 By a separation of variables find proper frequences of the system described by either (a) (*4 pts*) or (b) (*7 pts*):

(a) $$\begin{cases} u_{tt} + u_{xxxx} = 0, & -\pi < x < \pi, t > 0, \\ u|_{x=-\pi} = u_{xx}|_{x=-\pi} = u|_{x=\pi} = u_{xx}|_{x=\pi} = 0 \end{cases}$$

(b) $$\begin{cases} -(\Delta u)_t t = u_{zz}, 0 < x < \pi, 0 < y < \pi, 0 < z < \pi \\ u|_{x=0} = u|_{x=\pi} = u|_{y=0} = u|_{y=\pi} = u|_{z=0} = u|_{z=\pi} = 0 \end{cases}$$

MT-3 (*4 pts*) By a method of separation find a solution to

$$\begin{cases} u_{tt} - 4u_{xx} = 0, & -\pi < x < \pi, t > 0, \\ u_x|_{x=-\pi} = u_x|_{x=\pi} = 0, & \\ u|_{t=0} = x^2, & u_t|_{t=0} = 0 \end{cases}$$

MT-4 (*4 pts*) By a method of separation find a bounded solution to

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 & r > 1, \\ u|_{r=1} = |\sin\theta|. \end{cases}$$

MT-5 (*4 pts*) By a method of separation find a solution to

$$\begin{cases} u_{tt} - u_{xx} - u_{yy} = 0, & -\pi < x < \pi, 0 < y < \pi, \\ u|_{x=-\pi} = 0, & u|_{x=\pi} = 0, \\ u|_{y=0} = 0, & u|_{y=\pi} = 0, \\ u|_{t=0} = 0, & u_t|_{t=0} = \min(x, y). \end{cases}$$

APM 346, Mid-Term Test# 1 Solutions.

MT-1 (a) Equation yields that $u(x,t)=f(x+3t)+g(x-3t)$ as $x>0,t>0$. Initial condition imply that $f(x)+g(x)=\cos x$, $3f'(x)-3g'(x)=\sin x \implies f(x)-g(x)=-\frac{1}{3}\cos x$ and then $f(x)=\frac{1}{3}\cos x, g(x)=\frac{2}{3}\cos x$ as $x>0$. On the other hand, boundary condition implies that $f(3t)+g(-3t)=0 \implies g(x)=-f(-x)=-\frac{1}{3}\cos x$ as $x<0$ and finally $u(x,t)=\begin{cases}\dfrac{1}{3}\cos(x+3t)+\dfrac{2}{3}\cos(x-3t) & x>3t\\ \dfrac{1}{3}\cos(x+3t)-\dfrac{1}{3}\cos(x-3t) & 0<x<3t\end{cases}$

(b) Equations yiels that $u^+(x,t)=f(x+3t)+g(x-3t)$ as $x>0,t>0$ and $u^-(x,t)=h(x+t)+k(x-t)$ as $x<0,t>0$. Initial conditions yield that $f(x)+g(x)=0, 3f'(x)-3g'(x)=0$ and therefore $f(x)=g(x)=$ as $x>0$; similarly, initial conditions yield that $h(x)+k(x)=0, h'(x)-k'(x)=0$ and therefore $h(x)=k(x)=0$ as $x<0$. This would give us solutions for $x>3t$ and $x<-t$. Further, transmission conditions (at $x=0$) yield that $f(3t)+g(-3t)=h(t)+k(-t)$ and $f'(3t)+g'(-3t)-\cos t=h'(t)+k'(-t) \implies \frac{1}{3}f(3t)-\frac{1}{3}g(-3t)-\sin t=h(t)-k(-t)$ as $t>0$. In this system we know $f(3t)=0, k(-t)=0$ and we recover

$g(x)=\frac{3}{2}\sin\frac{x}{3}$ as $x<0$. Substituting into u we get

$$u^+(x,t)=\begin{cases}=0 & x>2t,\\ -\dfrac{3}{2}\sin(t-\dfrac{x}{3}) & 0<x<2t\end{cases}, \qquad u^-(x,t)=\begin{cases}0 & x<-t,\\ -\dfrac{3}{2}\sin(t+x) & 0>x-t\end{cases}.$$

MT-2 (a) Separating variables $u=X(x)T(t)$ we get

$$X^{IV}-\lambda X=0,\quad X(-\pi)=X(\pi)=0, X''(-\pi)=X''(\pi)=0,$$
$$T''+\omega^2T=0$$

and the first equation yields that $X=A\cosh kx+B\sinh kx+C\cos kx+D\sin kx$, $\lambda=k^4>0$; plugging into boundary conditions we recover $A\cosh k\pi=B\sinh k\pi=C\cos k\pi=D\sin k\pi$ which yields that either $k=n$, $X=\sin nx$ $(n=1,2,3,\dots)$ or $k=n-\frac{1}{2}$, $X=\cos(n-\frac{1}{2})x$ $(n=1,2,3,\dots)$.

So, $\omega=\frac{1}{4},1,\frac{9}{4},4\dots$.

(b) Separating variables $u=T(t)X(x)Y(y)Z(z)$ we get

$$\frac{T''}{T}\left(\frac{X''}{X}+\frac{Y''}{Y}+\frac{Z''}{Z}\right)=-\frac{Z''}{Z} \tag{1}$$

with boundary conditions

$$X(0)=X(\pi)=Y(0)=Y(\pi)=Z(0)=Z(\pi)=0 \tag{2}$$

Since right-side does not depend on t,x,y so does the left side. This leaves us with $T''=-\omega^2T$ ($Z''=0$ is not an option due to the boundary conditions), $X''=\mu X$, $Y''=\nu Y$ and then $Z''=\rho Z$ (with $\mu<0,\nu<0,\rho<0$ and we know solutions: $\mu=-m^2,\nu=-n^2,\rho=-p^2$ with $m,n,p=1,2,3,\dots$; so

$$\omega_{n,p,k}=\frac{k}{\sqrt{m^2+n^2+p^2}}$$

solves the problem.

MT-3 Separating variables we'll find that $X'' + \lambda X = 0$, $X'(-\pi) = X'(\pi) = 0$ which leads to $\lambda_n = \frac{1}{4}n^2$, $X_n = \cos\frac{n(x+\pi)}{2}$, $T_n'' + n^2 T_n = 0$, $T_n = A_n \cos nt + B_n \sin nt$, $X_0 = \frac{1}{2}$, $T_0 = A_0 + B_0 t$ and

$$u(x,t) = \frac{1}{2}(A_0 + B_0 t) + \sum_{n=1}^{\infty} \big(A_n \cos nt + B_n \sin nt\big) \cos\frac{n(x+\pi)}{2}.$$

Substituting into initial conditions we get $\frac{1}{2} + \sum_{n=1}^{\infty} A_n \cos\frac{n(x+\pi)}{2} = x^2$ and $B_n = 0$. The first equation yield that

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos\frac{n(x+\pi)}{2} x^2\,dx.$$

Integrating twice by parts we ses that

$$A_n = -\frac{8}{n^2\pi}(\cos n\pi - 1) = \begin{cases} 0 & n = 2m > 0 \\ \dfrac{16}{(2m+1)^2\pi} & n = 2m+1 \\ \dfrac{\pi^2}{3} & n = 0 \end{cases}$$

and therefore

$$u = \frac{\pi^2}{6} + \sum_{m=0}^{\infty} \frac{16}{(2m+1)^2\pi} \cos(m+\frac{1}{2})(x+\pi) =$$

$$\frac{\pi^2}{6} + \sum_{m=0}^{\infty} (-1)^{m+1} \frac{16}{(2m+1)^2\pi} \sin(m+\frac{1}{2})x.$$

MT-4 Separating variables we'll find that $\Theta'' + \lambda\Theta = 0$ and Θ is 2π-periodic which imply that

$$\lambda_0 = 0, \quad \Theta_0 = \frac{1}{2},$$
$$\lambda_n = n^2, \quad \Theta_{n,1} = \cos n\theta, \quad \Theta_{n,2} = \sin n\theta \qquad n = 1, 2, \ldots$$

and $r^2 R_n'' + rR_n' + n^2 R_n = 0$ which yields that $R_0 = A_0 + B_0 \log r$, $R_{n,k} = A_{n,k} r^n + B_{n,k} r^{-n}$, for $n = 1, 2, \ldots$, $k = 1, 2$. Since solution should be bounded we discard log and r^n terms. Then

$$u(r,\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta) r^{-n}.$$

Boundary condition yields that

$$\frac{A_0}{2} + \sum_{n=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta = |\sin\theta|$$

and since $|\sin\theta|$ is an even function equal to $\sin\theta$ on $(0,\pi)$, we conclude that $B_n = 0$ and

$$B_n = \frac{2}{\pi}\int_0^\pi \sin\theta\cdot\cos n\theta d\theta = \begin{cases} 0 & n = 2m+1 \\ \dfrac{4}{(4m^2-1)\pi} & n = 2m \end{cases}$$

and

$$u = -\frac{2}{\pi} + \sum_{m=1}^{\infty} \frac{4}{(4m^2-1)\pi} r^{-2m}\cos mx.$$

MT-5 Separating variables we get $X'' + \lambda X = 0$, $X(0) = X(\pi) = 0$, $Y'' + \mu Y = 0$, $Y(0) = Y(\pi) = 0$ and $T' + (\lambda+\mu)T = 0$ which implies that

$$\lambda_m = m^2, \quad X_m = \sin mx,$$
$$\mu_n = n^2, \quad Y_n = \sin ny,$$
$$T_{m,n} = A_{m,n}e^{(m^2+n^2)t}$$

and

$$u(x,y,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{m,n}e^{A}_{m,n}e^{(m^2+n^2)t}\sin mx \sin ny$$

and from initial condition

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{m,n}\sin mx\sin ny = \sin(xy)$$

we deduct that

$$A_{m,n} = \frac{4}{\pi^2}\int_{\mathcal{D}} \min(x,y)\sin(mx)\sin(ny)dxdy =$$
$$\frac{4}{\pi^2}\int_0^\pi dy\int_y^\pi y\sin(mx)\sin(ny)dx + \frac{4}{\pi^2}\int_0^\pi dx\int_x^\pi x\sin(mx)\sin(ny)dy$$

where $\mathcal{D} = \{0 < y < x < \pi\} \cup \{0 < x < y < \pi\}$. The first integral is $\frac{4}{\pi^2 m}\int_0^\pi y\sin(ny)(\cos my - \cos m\pi)dy$ and using trigonometric formula and integration by parts we get $\frac{4}{\pi}(-1)^{n+m}\frac{n^2}{nm(n^2-m^2)}$ $(n \neq m)$ and adding symmetric (with respect to m,n) term we get

$$A_{m,n} = \frac{4}{\pi}(-1)^{m+n}\frac{1}{mn} \quad (n \neq m)$$

while for $n = m$ we get $\frac{3}{2\pi m}$ and the final answer is

$$u(x,y,t) = \sum_{1\le m,n<\infty, m\neq n} \frac{4}{\pi}(-1)^{m+n}\frac{1}{mn}e^{-(m^2+n^2)t}\sin mx\sin ny +$$
$$\sum_{m=1}^{\infty}\frac{3}{2\pi m}e^{-2m^2t}\sin mx\sin my$$