

**APM 346F, GHA # 3. Nov. 8–Dec 1, 00**  
**Solutions**

1) 1+1 pts Find solution to

$$\begin{cases} u_t - u_{xx} = 0 & -\infty < x < \infty, t > 0 \\ u|_{t=0} = xe^{-|x|}, \end{cases}$$

(a) as a Fourier integral (don't try to calculate!).

(b) using Green's function. Express the answer using  $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-\frac{z^2}{2}} dz$ .

*Solution* (a) Making Fourier transform we get

$$\hat{u}_t + \omega^2 \hat{u} = 0$$

and therefore

$$\hat{u} = e^{-\omega^2 t} \hat{f}(\omega)$$

where  $f = u|_{t=0}$ . Therefore

$$(*) \quad u(x, t) = \int_{-\infty}^{\infty} e^{-\omega^2 t - i\omega x} \hat{f}(\omega) d\omega.$$

Calculating

$$\begin{aligned} \hat{f} &= \frac{1}{2\pi} \int_0^{\infty} x e^{-i\omega x} e^{-x} dx + \frac{1}{2\pi} \int_{-\infty}^0 x e^{-i\omega x} e^x dx = \frac{1}{2\pi} (1 + \omega i)^{-2} + \frac{1}{2\pi} (1 - \omega i)^{-2} = \\ & \qquad \qquad \qquad \frac{1}{\pi} (1 - \omega^2)(1 + \omega^2)^{-1} \end{aligned}$$

we get

$$u(x, t) = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-\omega^4 t - i\omega x} (1 - \omega^2)(1 + \omega^2)^{-2} d\omega.$$

(b) Using Green function  $G(x, y, t) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{(x-y)^2}{4t}}$  we get

$$\begin{aligned} u(x, t) &= \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} \times y e^{-|y|} dy = \\ &= \frac{1}{2\sqrt{\pi t}} \left( \int_0^{\infty} y e^{-\frac{(x-y)^2}{4t}} e^{-y} dy + \int_{-\infty}^0 y e^{-\frac{(x-y)^2}{4t}} e^y dy \right) = \\ &= \frac{1}{2\sqrt{\pi t}} \left( \int_0^{\infty} y e^{-\frac{(y-x+2t)^2}{4t} - x-t} dy + \int_{-\infty}^0 y e^{-\frac{(y-x-2t)^2}{4t} + x-t} dy \right) = \\ &= \frac{1}{2\sqrt{\pi t}} \left( \int_{2t-x}^{\infty} (z+x-2t) e^{-\frac{z^2}{4t} - x-t} dz + \int_{-\infty}^{-2t-x} (z+x+2t) e^{-\frac{z^2}{4t} + x-t} dz \right) = \\ &= \frac{1}{\sqrt{2}} (x-2t) \left( 1 - \Phi\left(\frac{2t-x}{\sqrt{2t}}\right) \right) e^{-x-t} + \frac{1}{\sqrt{2}} (x+2t) \Phi\left(\frac{-2t-x}{\sqrt{2t}}\right) e^{x-t} \end{aligned}$$

**2** Find solution to discrete Airy equation (don't try to calculate!) as a Fourier integral:

$$\frac{1}{h^2}(u(x+h) - 2u(x) + u(x-h)) + xu = 0$$

where  $h$  is a constant.

*Solution* Making Fourier transform we get

$$h^{-2}(e^{ih\omega} - 2 + e^{-ih\omega})\hat{u} + i\frac{d}{d\omega}\hat{u} = 0$$

because of the Fourier transform of the shifted function formula, or

$$-2h^{-2}(1 - \cos \omega h)\hat{u} + i\frac{d}{d\omega}\hat{u} = 0$$

which implies

$$\hat{u} = C e^{ih^{-3}(\omega h - \sin(\omega h))}$$

and

$$u = C \int_{-\infty}^{\infty} e^{ih^{-3}(\omega h - \sin(\omega h)) + i\omega x}$$

(other linearly independent solutions of the original equation grows quickly at  $-\infty$  and is filtered out by a Fourier transform method).

**3) 1+1 pts** Find solution to Laplace equation in the half-plane

$$\begin{cases} u_{xx} + u_{yy} = 0 & -\infty < x < \infty, x > 0 \\ u|_{x=0} = ye^{-y^2} \end{cases}$$

(a) as a Fourier integral (don't try to calculate!);

(b) using Green's function (don't try to calculate!).

*Solution* (a) Making Fourier transform with respect to  $y$  we get

$$\hat{u}_{xx} - \omega^2 \hat{u} = 0, \quad \hat{u}|_{y=0} = \frac{-2i\omega}{\pi(\omega^2 + 1)^2}$$

and from equation it follows that  $\hat{u} = A(\omega)e^{\omega y} + B(\omega)e^{-\omega y}$ . Discarding the first term as exponentially large (so,  $A(\omega) = 0$ ) and finding  $B(\omega) = \frac{-2i\omega}{\pi(\omega^2 + 1)^2}$  we get  $\hat{u} = \frac{-2i\omega}{\pi(\omega^2 + 1)^2} e^{-\omega y}$  and

$$u(x, y) = -2i \int_{-\infty}^{\infty} \frac{\omega}{\pi(\omega^2 + 1)^2} e^{\omega(-y+ix)} d\omega$$

(b) Starting from logarithmic potential  $-\frac{1}{2\pi} \log \rho$  we get a Green function on the whole plane

$$G^0(x, y; x', y') = -\frac{1}{4\pi} \log((x - x')^2 + (y - y')^2)$$

and the Green function for the whole plane (with the boundary condition above):

$$G(x, y; x', y') = -\frac{1}{4\pi} \log((x - x')^2 + (y - y')^2) - \frac{1}{4\pi} \log((x + x')^2 + (y - y')^2)$$

and then

$$u(x, y) = -\int_{-\infty}^{\infty} \frac{\partial}{\partial y'} G(x, y; 0, y') \times f(y') dy' = \frac{1}{\pi} \int_{-\infty}^{\infty} ((y - y')^2 + x^2)^{-1} y' e^{-|y'|} dx'$$

4) 1 pt Find solution to the heat equation in the annulus

$$\begin{cases} u_t - u_{xx} - u_{yy} = 0 & -\infty < x, y < \infty, t > 0 \\ u|_{t=0} = \sin(x^2 + y^2) \end{cases}$$

Find  $u(0, 0, t)$  and maximum of it.

*Solution* Using Green function

$$G(x, y; x', y'; t) = \frac{1}{4\pi t} e^{-\frac{1}{4t}((x-x')^2+(y-y')^2)}$$

one can express solution

$$u(x, y, t) = \frac{1}{4\pi t} \iint e^{-\frac{1}{4t}((x-x')^2+(y-y')^2)} f(x', y') dx' dy'$$

with  $f = u|_{t=0}$  and for  $x = y = 0$  in the polar coordinates

$$u(x, y, t) = \frac{1}{4\pi t} \int_0^{2\pi} d\theta \int_0^{\infty} e^{-\frac{\rho^2}{4t}} f(x', y') \rho d\rho = \frac{1}{2t} \int_0^{\pi} e^{-\frac{z}{4t}} \sin z dz = \frac{8t}{1 + 16t^2}$$

which reaches maximum for  $t = \frac{1}{4}$ .

5) 1 pt Find solution to

$$u_{xx} + u_{yy} + u_{zz} = \begin{cases} 1 & x^2 + y^2 < 1, -\frac{3}{4} < z < \frac{3}{4} \\ 0 & \text{otherwise} \end{cases}, \quad |u| \rightarrow 0 \text{ as } x^2 + y^2 + z^2 \rightarrow \infty$$

in the whole space (don't try to calculate!). Calculate  $u(0, 0, 0)$ .

*Solution* Using Coulomb potential

$$u(x, y, z) = -\frac{1}{4\pi} \iiint \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} f(x', y', z') dx' dy' dz'$$

with  $f = \Delta u$  we get

$$u(0, 0, 0) = -\frac{1}{4\pi} \iiint_{\{x^2+y^2 < 1, -\frac{3}{4} < z < \frac{3}{4}\}} dx dy dz = -\frac{1}{2} \int_{-1}^1 dz \int_0^1 \sqrt{\rho^2 + z^2} \rho d\rho$$

where we used cylindrical coordinates and integrated with respect to  $\phi$

$$= - \int_0^{\frac{3}{4}} (\sqrt{1+z^2} - z) dz = -\frac{1}{2} (z\sqrt{1+z^2} + \log(z + \sqrt{1+z^2}) - z^2) \Big|_{z=0}^{z=\frac{3}{4}} = -\frac{3}{16} - \frac{1}{2} \log 2$$

6) 1 pt Find (don't calculate!) a solution using Green function for

$$\begin{cases} u_{\rho\rho} + \frac{1}{\rho}u_{\rho} + \frac{1}{\rho^2}u_{\theta\theta} = 0, & \rho < 3, \\ u|_{\rho=3} = |\sin \theta| \end{cases}$$

Calculate  $u$  at  $(0, 0)$ .

**Solution** Using Poisson formula

$$u(\rho, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{9 - \rho^2}{9 - 6\rho \cos(\theta - \phi) + \rho^2} |\sin \phi| d\phi$$

Then

$$u(0, 0) = \frac{1}{2\pi} \int_0^{2\pi} |\sin \phi| d\phi = \frac{2}{\pi}.$$

7) 1 pt Find Green function for

$$\begin{cases} u_t - u_{xx} - u_{yy} = 0, & t > 0, x > 0, \sqrt{3}x > y > 0 \\ u|_{x=0} = 0, \\ u|_{y=0} = 0 \end{cases}$$

**Note** There was a misprint:  $\frac{\sqrt{3}}{2}$  in original settings. I am asking TA to mark problem taking this misprint into account *Solution* Starting from the Green function for the whole plane

$$G_0(x, y; x', y'; t) = \frac{1}{4\pi t} e^{-\frac{1}{4t}((x-x')^2 + (y-y')^2)}$$

and using mirror symmetry method we get

$$G(x, y; x', y'; t) = \sum_{i=0}^2 G_0(x, y; T^i(x', y'); t) - \sum_{i=0}^2 G_0(x, y; T^i(x', -y'); t)$$

where  $T = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$  is the matrix of  $\frac{2\pi}{3}$ -rotation.

8) 1 pt Calculate  $u(0, 0, 0, t)$  where  $u$  is a solution to

$$\begin{cases} u_{tt} - u_{xx} - u_{yy} - u_{zz} = 0, \\ u|_{t=0} = 0, \quad u_t|_{t=0} = \begin{cases} \sqrt{4 - x^2 - y^2 - z^2} & 0 < x^2 + y^2 + z^2 < 4 \\ 0 & \text{otherwise} \end{cases} \end{cases}$$

*Solution* We know that

$$u(x, y, z, t) = \frac{\partial}{\partial t} \left( \frac{1}{4\pi t} \iint_{S_t(x, y, z)} f(x', y', z') dS \right)$$

where  $f = u_t|_{t=0}$  (because  $u_t|_{t=0} = 0$ ) and  $S_t(x, y, z)$  is a sphere of the radius  $|t|$  with a center at  $(x, y, z)$ . For  $(x, y, z) = (0, 0, 0)$  his integral is equal to  $4\pi t^2 \sqrt{4 - t^2}$  for  $|t| \leq 2$  and 0 otherwise and

$$u(0, 0, 0, t) = \begin{cases} \frac{\partial}{\partial t} (t\sqrt{4 - t^2}) = \frac{4 - 2t^2}{\sqrt{4 - t^2}}, & |t| < 2. \\ 0 & |t| > 2 \end{cases}$$