APM 346 GHA #2 Solutions (Fall 2000)

2-1) *1.5 pt* Solve the problems below:

$$\begin{cases} u_{tt} - 16u_{xx} = f(x,t), \quad (t > 0, 0 < x < 2) \\ u|_{x=0} = u|_{x=2} = 0, \\ u|_{t=0} = g(x), \quad u_t|_{t=0} = h(x) \end{cases}$$

with

(a)	$f = 0,$	$g = \sin(\frac{\pi x}{2}),$	$h = -\sin(\pi x)$
(b)	$f = 0,$	$g = 1 - \|1 - x\|,$	$h = 0$
(c)	$f = \sin(\pi x)\sin(2\pi t),$	$g = 0,$	$h = 0$
(d)	$f = \sin(\frac{\pi x}{2})\sin(2\pi t),$	$g = 0,$	$h = 0$

Solution Separation of variables leads to

$$\text{(1)} \qquad \lambda_n = -\left(\frac{\pi n}{2}\right)^2, \qquad X_n = \sin\frac{\pi n x}{2}$$
$$T_n(t) = A_n\cos(2\pi n t) + B_n\sin(2\pi n t),$$
$$u(x,t) = \sum_{n=1}^{\infty}\big(A_n\cos(2\pi n t) + B_n\sin(2\pi n t)\big)\sin\frac{\pi n x}{2}$$

where second and third lines are valid only as $f = 0$.

(a) Both functions g, h are decomposed in appropriate functions X_n already $n = 1, 2$ and $A_1 = 1, B_1 = 0$ while $A_2 = 0, \pi B_2 = -\frac{1}{4\pi}$;

$$u = \sin(2\pi t)\sin\frac{\pi x}{2} - \frac{1}{4\pi}\sin(4\pi t)\sin(\pi x)$$

(b) From initial conditions we have $B_n = 0$ and A_n are coefficients of decomposition $g(x)$ into X_n-series: $A_n = \int_0^2 (1 - |1-x|)\sin\frac{\pi n x}{2}dx$. Since $(1 - |1-x|)$ is even with respect to $x = 1$ (center of $(0,2)$) and $\sin\frac{\pi n x}{2}$ are even/odd for odd/even n, $A_{2m} = 0$,

$$A_{2m+1} = 4\int_0^1 x\sin\frac{\pi(2m+1)x}{2}dx =$$
$$\left(-x\cdot\frac{2}{\pi(2m+1)}\cos\frac{\pi(2m+1)x}{2} + \frac{4}{\pi(2m+1)^2}\sin\frac{\pi(2m+1)x}{2}\right)\Big|_0^1 = \frac{4}{\pi(2m+1)^2}(-1)^m;$$
$$u(x,t) = \sum_{m=0}^{\infty}\frac{4}{\pi(2m+1)^2}(-1)^m\cos(2(2m+1)\pi n t)\sin\frac{\pi(2m+1)x}{2}$$

In (c),(d) we take

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin\frac{\pi n x}{2}$$

and obviously $T_n = 0$ unless $n = 2$ in (c), $n = 1$ in (d) and

$$T_n'' + (2\pi n)^2 T_n = \sin(2\pi t), \quad T_n(0) = 0, T_n'(0) = 0$$

in both cases.

(c) Solving ordinary equation we get $T_2 = -\frac{1}{12\pi^2}\sin(2\pi t) + A\cos(4\pi t) + B\sin(4\pi t)$; initial conditions imply $A = 0$, $B = \frac{1}{24\pi^2}$ and $u = \bigl(-\frac{1}{12\pi^2}\sin(2\pi t) + \frac{1}{24\pi^2}\sin(4\pi t)\bigr)\sin \pi x$.

(d) Solving ordinary equation we get $T_2 = -\frac{1}{4\pi}t\cos(2\pi t) + A\cos(2\pi t) + B\sin(2\pi t)$; initial conditions imply $A = 0$, $B = \frac{1}{8\pi^2}$, and $u = \bigl(-\frac{1}{4\pi}t\cos(2\pi t)\sin(2\pi t) + \frac{1}{8\pi^2}\sin(2\pi t)\bigr)\sin\frac{\pi x}{2}$.

2-2 *1.5 pt* Solve the problems below and find the limits of u as $t \to \infty$:

$$\begin{cases} u_t - 9u_{xx} = f(x,t), & (t > 0, 0 < x < \pi) \\ u|_{x=0} = u|_{x=\pi} = 0, \\ u|_{t=0} = g(x), \end{cases}$$

with

(a)	$f = 0,$	$g = \sin x,$
(b)	$f = 0,$	$g = 1,$
(c)	$f = \sin 2x,$	$g = 0,$
(d)	$f = x(\pi - x),$	$g = 0,$

Solution Separation of variables leads to

(2)
$$\begin{aligned} &\lambda_n = -n^2, \qquad X_n = \sin nx \\ &T_n(t) = e^{-9n^2 t}, \\ &u(x,t) = \sum_{n=1}^{\infty} A_n e^{-9n^2 t}\sin nx \end{aligned}$$

where second and third lines are valid only as $f = 0$.

(a) $A_n = 0$ unless $n = 1$, $A_1 = 1$, $u = e^{-9t}\sin x$, $u \to 0$ as $t \to +\infty$.

(b) From initial condition A_n are coefficients of decomposition of 1 into X_n series: $A_{2m} = 0$, $A_{2m+1} = \frac{4}{(2m+1)\pi}$ and $u = \sum_{m=0}^{\infty}\frac{4}{(2m+1)\pi}e^{-9n^2 t}\sin nx$, $u \to 0$ as $t \to +\infty$.

In (c),(d)

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin nx$$

where T_n satisfy

$$T_n' + 9n^2 T_n = F_n, \qquad T_n(0) = 0$$

with F_n coefficients of decomposition of f into X_n-series.

(c) $T_n = 0$ unless $n = 2$, $F_2 = 1$ and $T_2' + 36T_2 = 1$, $T_2 = \frac{1}{36}(1 - e^{-36t})$, $u = \frac{1}{36}(1 - e^{-36t})\sin 2x$, $u \to \frac{1}{36}\sin 2x$ as $t \to +\infty$.

(d)

$$F_n = \frac{2}{\pi}\int_0^\pi x(\pi - x)\sin nx dx = \frac{2}{\pi n}\int_0^\pi (\pi - 2x)\cos nx dx = -\frac{4}{\pi n^3}\cos nx\Big|_{x=0}^{x=\pi}$$

and $F_{2m} = 0$, $F_{2m+1} = \frac{8}{\pi(2m+1)^3}$. Therefore $T_{2m} = 0$, $T_{2m+1} = \frac{8}{\pi 9}(2m+1)^5(1 - e^{-9(2m+1)^2 t})$, $u = \sum_{m=0}^\infty \frac{8}{\pi 9(2m+1)^5}(1 - e^{-9(2m+1)^2 t})\sin(2m+1)x$, $u \to \sum_{m=0}^\infty \frac{8}{\pi 9(2m+1)^5}\sin(2m+1)x$ as $t \to +\infty$.

2-3) *1.5 pt* Solve the problems below:

$$\begin{cases} u_{xx} + 4u_{yy} = f(x,t), & (0 < x < \pi, 0 < y < \pi) \\ u|_{x=0} = u|_{x=\pi} = 0, \\ u|_{y=0} = g(y), & u_y|_{y=\pi} = h(x) \end{cases}$$

with

(a)	$f = 0,$	$g = \sin x,$	$h = -\sin 2x$
(b)	$f = 0,$	$g = x,$	$h = 0$
(c)	$f = \sin x \cdot \sin \frac{y}{2},$	$g = 0,$	$h = 0$
(d)	$f = 1,$	$g = 0,$	$h = 0$

Solution Separation of variables leads to $\lambda_n = -n^2$, $X_n = \sin nx$ $(n = 1, 2, \dots)$ and in cases (a),(b) to $Y_n = A_n e^{ny} + B_n e^{-ny}$.

(a) In this case $Y_n = 0$ unless $n = 1, 2$ and to satisfy boundary condition we need to take

$$Y_2 = -\frac{\sinh 2y}{2\cosh 2\pi}, \quad Y_1 = \frac{\cosh(\pi - y)}{\cosh \pi}, \qquad u = \frac{\cosh(\pi - y)}{\cosh \pi}\sin x - \frac{\sinh 2y}{2\cosh 2\pi}\sin 2x.$$

(b) In this case $Y_n = C_n \frac{\cosh n(\pi - y)}{\cosh n\pi}$ where C_n are coefficients in X_n-series of g:

$$C_n = \frac{2}{\pi}\int_0^\pi x \sin nx dx = \frac{2}{\pi n}(-1)^{n+1};$$

$$u = \sum_{n=1}^\infty \frac{2}{\pi n}(-1)^{n+1}\frac{\cosh n(\pi - y)}{\cosh n\pi}\sin nx.$$

(c) In this case $Y_n = 0$ unless $n = 1$ and $u = Y_1(y)\sin x$ where Y_1 solves the problem

$$Y_1'' - Y_1 = \sin\frac{y}{2}, \qquad Y_1(0) = Y_1'(\pi) = 0$$

and therefore $Y_1 = -\frac{4}{5}\sin\frac{y}{2}$,

$$u = -\frac{4}{5}\sin\frac{y}{2}\sin x$$

(since $\sin\frac{y}{2}$ is an eigenfunction of the problem with respect to y).

(d) In this case we need to decompose first f with respect to X_n: $f = \sum F_n(y)\sin nx$ with $F_{2m} = 0$, $F_m = \frac{4}{\pi(2m+1)}$ and therefore $Y_{2m} = 0$ and

$$Y''_{2m+1} - (2m+1)^2 Y_{2m+1} = \frac{4}{\pi(2m+1)} \implies$$
$$Y_{2m+1} = -\frac{4}{\pi(2m+1)^3} + C_m e^{(2m+1)y} + D_m e^{-(2m+1)y}$$

where satisfying boundary conditions gives us

$$Y_{2m+1} = -\frac{4}{\pi(2m+1)^3}(1 - \frac{\cosh(2m+1)(\pi - y)}{\cosh(2m+1)\pi},$$

and

$$u = \sum_{m=0}^{\infty} -\frac{4}{\pi(2m+1)^3}(1 - \frac{\cosh(2m+1)(\pi - y)}{\cosh(2m+1)\pi}\sin(2m+1)x.$$

2-4) *1.5 pt* Solve the problems (in polar coordinates) below:

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = f(r,\theta), & (\frac{1}{2} < r < 2, 0 < \theta < 2\pi) \\ u|_{r=\frac{1}{2}} = g(\theta), \qquad u|_{r=2} = h(\theta) \end{cases}$$

with

(a) $f = 0,\qquad g = 1,\qquad h = \sin 2\theta$

(b) $f = 0,\qquad g = \begin{cases} 1 & 0 < \theta < \pi \\ 0 & \pi < \theta < 2\pi \end{cases},\qquad h = \begin{cases} -1 & 0 < \theta < \pi \\ 0 & \pi < \theta < 2\pi \end{cases}$

(c) $f = r,\qquad g = 0,\qquad h = 0$

(d) $f = \sin 2\theta,\qquad g = 0,\qquad h = 0$

Solution Separating variables: $u(x,y) = \Theta(\theta)R(r)$ we get $\Theta'' - \lambda\Theta = 0$ and Θ must be 2π-periodic: $\lambda_0 = 0, \Theta_0 = \frac{1}{2}$, $\lambda_n = -n^2, \Theta_{n1} = \cos n\theta, \Theta_{n2} = \sin n\theta$ $(n = 1, \dots)$. Further $R_0 = A_0 + B_0 \log r$, $R_{nj} = A_{nj}r^n + B_{nj}r^{-n}$ $(n = 1, \dots, j = 1, 2)$ (as $f = 0$)

(a) $R_{nj} = 0$ unless $n = 0$ or $n = 2, j = 2$; $\frac{1}{2}R_0$ should be equal 1 as $r = \frac{1}{2}$ and 0 as $r = 2$ and therefore $R_0 = \frac{\log\frac{r}{2}}{\log 2}$. Further. $R_{22} = 0$ as $r = \frac{1}{2}$ and $R_{22} = 1$ as $r = 2$ and therefore $R_{22} = \frac{4}{255}(16r^2 - r^{-2})$:

$$u = \frac{1}{2}\frac{\log\frac{r}{2}}{\log 2} + \frac{4}{255}(16r^2 - r^{-2})\sin 2\theta.$$

(b)

$$u = \frac{1}{2}R_0 + \sum_{n=1}^{\infty}\bigl(R_{n1}\cos n\theta + R_{n2}\sin n\theta\bigr).$$

Substituting to boundary conditions:

$$\frac{1}{2}R_0 + \sum_{n=1}^{\infty}\bigl(R_{n1}\cos n\theta + R_{n2}\sin n\theta\bigr)\bigr|_{r=\frac{1}{2}} = g(\theta),$$

$$\frac{1}{2}R_0 + \sum_{n=1}^{\infty}\bigl(R_{n1}\cos n\theta + R_{n2}\sin n\theta\bigr)\bigr|_{r=2} = h(\theta);$$

Decomposing $g(\theta)$ into Fourier series:

$$g(\theta) = \frac{1}{2} + \sum_{m=0}^{\infty}\frac{4}{\pi(2m+1)}\sin(2m+1)\theta$$

and noting that $h = -g$ we understand that

- $R_0(\frac{1}{2}) = -R_0(2) = \frac{1}{2} \implies R_0 = -\frac{\log r}{\log 2}$; • $R_{nj} = 0$ unless n is odd and $j = 2$;
- $R_{(2m+1)j} = C_m(r^{2m+1} - r^{-(2m+1)r})$ and $C_m(2^{2m+1} - 2^{-2(2m+1)}) = -\frac{4}{\pi(2m+1)}$

which leads to a solution

$$u = -\frac{1}{2}\frac{\log r}{\log 2} - \sum_{m=0}^{\infty} - \frac{4}{\pi(2m+1)(2^{2m+1} - 2^{-2(2m+1)})}(r^{2m+1} - r^{-(2m+1)r})\sin(2m+1)\theta$$

(c) In this case $R_{nj} = 0$ for $n \geq 1$ and $u = v(r)$ only: $v'' + \frac{1}{r}v' = r$, $v(\frac{1}{2}) = v(2) = 0$. Equation can be rewritten as $(v'r)' = v''r + v' = r^2$ which implies $v = \frac{1}{9}r^3 + A + B\log r$. To satisfy boundary conditions we must take $A = -\frac{257}{144}$, $B = -\frac{255}{144\log 2}$:

$$u = \frac{1}{9}r^3 - \frac{257}{144} - \frac{255}{144\log 2}\log r.$$

(d) Similarly, $u = v(r)\sin 2\theta$ with

$$v'' + \frac{1}{r}v' - \frac{4}{r^2}v = 1, \quad v(\frac{1}{2}) = v(2) = 0.$$

Pluging $v = Cr^2$ we get 0, pluging $v = Cr^2\log r$ we get $6C$; so general solution is $\frac{1}{6} + Ar^2 + Br^{-2}$ and to satisfy boundary conditions we must take $A = B = -\frac{2}{51}$. So,

$$u = \bigl(\frac{1}{6} - \frac{2}{51}r^2 - \frac{2}{51}r^{-2}\bigr)\sin 2\theta.$$

2-5) *1.5 pt* Solve the problems below:

$$\begin{cases} u_{tt} - u_{xx} - u_{yy} = f(x,y,t), \quad (0 < x < \pi, 0 < y < 2\pi) \\ u_x|_{x=0} = u_x|_{x=\pi} = 0, \\ u|_{y=0} = u|_{y=2\pi} = 0, \\ u|_{t=0} = g(x,y), \quad u_t|_{t=0} = h(x,y) \end{cases}$$

with

(a)	$f = 0,$	$g = 0,$	$h = \cos x \sin y$
(b)	$f = 0,$	$g = 0$	$h = \sin x \cos y$
(c)	$f = \cos 3x \sin 4y,$	$g = 0,$	$h = 0$
(d)	$f = \cos 3x \sin 4y \cos 5t,$	$g = 0,$	$h = 0$

Solution Separating variables $u = X(x)Y(y)T(t)$ we get $X_0 = \frac{1}{2}$, $X_m = \cos\frac{mx}{2}$, $(m = 1, \dots)$ $Y_n = \sin\frac{ny}{2}$ $(n = 1, 2, \dots)$, $T_{mn} = A_{mn}\cos\omega_{mn}t + B_{mn}\sin\omega_{mn}t$, $\omega_{mn} = \frac{1}{2}\sqrt{m^2+n^2}$. $m = 0, 1, \dots,$ $n = 1, 2, \dots$.

(a) In this case $T_{mn} = 0$ unless $m = 2$, $n = 2$ and $T_{22} = \frac{1}{\sqrt{2}}\sin\sqrt{2}t$, $u = \frac{1}{\sqrt{2}}\sin\sqrt{2}t\cos x\sin y$.

(b) In this case $A_{mn} = 0$ and

$$\omega_{mn}B_{mn} = \frac{1}{\pi^2}\int_0^{2\pi}\int_0^{2\pi}\sin x\cos y\cos\frac{mx}{2}\sin\frac{ny}{2}dxdy$$

and calculations show that all coefficients vanish but for

$$B_{2p+1,2q+1} = \frac{8}{\pi^2(4p^2+4p-3)(4q^2+4q-3)\sqrt{(2p+1)^2+(2q+1)^2}}.$$

Plug by yourseves.

(c),(d) In this cases $u = T(t)\cos 3x\sin 4y$ and

$$T'' + 25T = F, \quad T(0) = T'(0) = 0$$

with $F = 1$ in (c), $F = \cos 5t$ in (d). Then in (c) $T = \frac{1}{25}(1-\cos 5t)$ and $u = \frac{1}{25}(1-\cos 5t)\cos 3x\sin 4y$ while in (d) $T = \frac{1}{10}t\sin 5t$ (resonance!) and $u = \frac{1}{10}t\sin 5t\cos 3x\sin 4y$.

2-6) *1.5 pt* Find equation for resonance frequences of the annulus:

$$\begin{cases} u_{tt} - u_{rr} - \frac{1}{r}u_r - \frac{1}{r^2}u_{\theta\theta} = 0, & (1 < r < 2, 0 < \theta < 2\pi) \\ u_r|_{r=1} = u_r|_{r=3} = 0, \end{cases}$$

Solution Separating variables $u = \Theta(\theta)R(r)T(t)$ we get

- $\Theta'' - \lambda\Theta = 0$ and Θ is 2π-periodic (so $\lambda = -n^2$, $n = 0, 1, \dots$.
- $T'' + \omega^2 T = 0$ where ω are eigenfrequencies
- $R'' + \frac{1}{r}R' + (\omega^2 - n^2r^{-2})R = 0$, $R(1) = R(3) = 0$.

Pluging $\omega r = s$ we get similar equation with $\omega = 1$ and this is Bessel equation with Bessel functions solutions $AJ_n(\omega r) + BJ_{-n}(\omega r)$ where J_k are Bessel functions. Plugging into boundary conditions we get a homogeneous linear system and $A = B = 0$ unless

$$\begin{vmatrix} J_n(\omega) & J_{-n}(\omega) \\ J_n(3\omega) & J_{-n}(3\omega) \end{vmatrix} = 0$$

which is an equation in question.