APM 346F, GHA # 1. Sept. 15- Oct. 2, 2000
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1) *1pt* Find (a) The general solution to equation $u_t + 3u_x = 0$ and (b) solution, satisfying $u|_{t=0} = e^{-|x|}$; (c) The general solution to equation $u_t + u_x - u_y = 0$ and (b) Solution satisfying $u|_{t=0} = xy$

Solution. (a) Characteristics are $\frac{dt}{1} = \frac{dx}{3}$ or $x - 3t = C$. So $u = f(x - 3t)$ with arbitrary function (of one variable) f.

(b) From initial condition $f(x) = e^{-|x|}$ and $u = e^{-|x-3t|}$.

(c) Characteristics are $\frac{dt}{1} = \frac{dx}{1} = \frac{dy}{-1}$ or $x - t = C_1$, variables) f.

(d) From initial condition $f(x) = xy$ and $u = (x - t)(y + t)$.

2) *1pt* Find (a) The general solution to $u_t + tu_x = 0$ and (b) solution, satisfying $u|_{t=0} = x^2$.

Solution (a) Similarly, cCharacteristics are $\frac{dt}{1} = \frac{dx}{t}$ or $dx = -tdt$ or $x - \frac{t^2}{2} = C$ and $u = f(x - \frac{t^2}{2})$.

(b) From initial condition $f(x) = x^2$ and $u = (x - \frac{t^2}{2})^2$.

3) *1pt* Find (a) The general solution to $u_t + 3u_x = t - x$ and (b) solution satisfying $u|_{t=0} = 0$.

Solution. (a) Extended characteristic equation is $\frac{dt}{1} = \frac{dx}{3} = \frac{du}{t-x}$ or $x - 3t = C$ and $du = (-2t - C)dt \implies u = -t^2 - Ct + D$. So, the general solution is $u = -t^2 - (x - 3t)t + f(x - 3t) = 2t^2 - xt + f(x - 3t)$.

(b) From initial condition $f(x) = 0$ and $u = 2t^2 - xt$.

4) *1pt* In the domain $x > 0, t > 0$ solve $u_t - 3u_x = 0$ with initial condition $u|_{t=0} = \sin x$ (as $x > 0$) and with the boundary condition $u|_{x=0} = 0$ (as $t > 0$) (?) if equation and initial condition does not provide uniqueness of the solution (you *must* indicate this).

Solution. Characteristics are $\frac{dt}{1} = \frac{dx}{-3}$ or $x + 3t = C$. General solution to equation is $u = f(x + 3t)$. These characteristics, starting from $t = 0$, $x > 0$ cover the whole quadrant $t > 0, x > 0$ and *no boundary condition is needed*; $u = \sin(x + 3t)$.

5) *1pt* In the domain $x > 0, t > 0$ solve $u_t + 3u_x = 0$ with initial condition $u|_{t=0} = \sin x$ (as $x > 0$) and with the boundary condition $u|_{x=0} = 0$ (as $t > 0$) (?) if equation and initial condition does not provide uniqueness of the solution (you *must* indicate this.

Solution. Characteristics are $\frac{dt}{1} = \frac{dx}{3}$ or $x - 3t = C$. General solution to equation is $u = f(x - 3t)$. These characteristics, starting from $t = 0$, $x > 0$ cover sector $x > 3t > 0$; characteristics, starting from $x = 0, t > 0$ cover the rest.

Initial condition yields $f = \sin x$ $(x > 0)$ and boundary condition yields $f(x) = 0$ for $x < 0$. So

$$u = \begin{cases} \sin(x - 3t) & x > 3t > 0 \\ 0 & 03t > x > 0. \end{cases}$$

6) *1pt* For the system

$$(*) \qquad \begin{cases} u_t - 3u_x + 4v_x = 0, \\ v_t + 4u_x + 3v_x = 0 \end{cases}$$

(a) find the canonical form (Riemannian invariants)
(b) find general solution
(c)find solution of the Cauchy problem $u|_{t=0} = 0, v|_{t=0} = \cos x$.

Solution. One can rewrite sytem as $U_t + AU_x = 0$ with $U = \begin{pmatrix} u \\ v \end{pmatrix}$, $A = \begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}$.

Characteristic equation $\begin{vmatrix} -3-\lambda & 4 \\ 4 & 3-\lambda \end{vmatrix} = \lambda^2 - 25 = 0$ and $\lambda_{1,2} = \pm 5$.

Finding left eigenvectors: (L_1): $\begin{pmatrix} \alpha & \beta \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix} = 0$, $L_1 = \begin{pmatrix} 2 & -1 \end{pmatrix}$.

Similarly, $L_2 = \begin{pmatrix} 2 & 1 \end{pmatrix}$. So, $w_1 = L_1 U = 2u - v$, $w_2 = L_2 U = 2u + v$ and $w_{1t} - 5w_{1x} = 0$, $w_{2t} + 5w_{2x} = 0$; therefore $w_1 = f(x + 5t)$, $w_2 = g(x - 5t)$.
(b) From (a) $2u - v = f(x+5t)$, $2u + v = g(x-5t)$ and therefore $u = \frac{1}{4}\bigl(f(x+5t) + g(x-5t)\bigr)$, $u = \frac{1}{2}\bigl(-f(x+5t) + g(x-5t)\bigr)$.
(c) From initial conditions $\frac{1}{4}(f(x) + g(x)) = 0$, $\frac{1}{2}(-f(x) + g(x)) = \cos x$ and $g(x) = \cos x$, $f(x) = -\cos x$; finally $u = \frac{1}{4}\bigl(-\cos(x+5t) + \cos(x-5t)\bigr)$, $v = \frac{1}{2}\bigl(\cos(x+5t) + \cos(x-5t)\bigr)$.

7) *1pt* For the system (*) in $x > 0, t > 0$ with the initial conditions $u|_{t=0} = f(x), v|_{t=0} = g(x)$ (as $x > 0$) determine which boundary conditions are well- (or ill) posed:
(a) -;
(b) $u|_{x=0} = \phi(t)$ (as $t > 0$);
(c) $u|_{x=0} = \phi(t)$, $v|_{x=0} = \psi(t)$ (as $t > 0$);
(d) $(2u + v)|_{x=0} = \phi(t)$ (as $t > 0$);
(e) $(u - 2v)|_{x=0} = \phi(t)$ (as $t > 0$).
What happens if we replace "$x > 0$" by "$x < 0$" everywhere in this problem?

Solution Since w_1 is "incoming" to $x = 0$ ($t > 0$), and w_2 is "outgoing", we need exactly one boundary condition. So (a) and (c) are ill-posed (wrong number of conditions).
To define w_2 (at $x = 0$, $t > 0$) we have equation
(b) $\frac{1}{4}(-w_1 + w_2) = \phi$ (we can find w_2),
(d) $w_2 = \phi$ (we can find w_2),
(e) $w_1 = \phi$ (we cannot find w_2).
So, (b),(d) are well-posed, (a),(c),(e) are ill-posed.

8) *1pt*

$$\begin{cases} u_t + u_x - 6v_x + w_x = 0 \\ v_t - 5u_x + \quad w_x = 0 \\ w_t + 5u_x + 6v_x + 5w_x = 0 \end{cases}$$

(a) find the canonical form (Riemannian invariants)
(b) find general solution
(c) find solution of the Cauchy problem $u|_{t=0} = 0, v|_{t=0} = \cos x, w|_{t=0} = 0$.

Solution A in 7), $A = \begin{pmatrix} 1 & -6 & 1 \\ -5 & 0 & 1 \\ 5 & 6 & 5 \end{pmatrix}$, characteristic equation is $\begin{vmatrix} 1-\lambda & -6 & 1 \\ -5 & -\lambda & 1 \\ 5 & 6 & 5-\lambda \end{vmatrix} = -(6-\lambda)^2(6+\lambda) = 0$ and $\lambda_1 = \lambda_2 = 6$ is a double root, $\lambda_3 = -6$. Finding eigenvectors

(L_1, L_2):

$$\begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix} \begin{pmatrix} -5 & -6 & 1 \\ -5 & -6 & 1 \\ 5 & 6 & -1 \end{pmatrix} = 0 \iff -5\alpha - 6\beta + \gamma = 0.$$

We have one equation only, picking $\alpha = 1, \beta = 0$ and $\alpha = 0, \beta = 1$ we get two eigenvectors $L_1 = \begin{pmatrix} 1 & 0 & 5 \end{pmatrix}$, $L_2 = \begin{pmatrix} 0 & 1 & 6 \end{pmatrix}$. (we would be in trouble if egenspace was 1-dimensional)

L_3:

$$\begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix} \begin{pmatrix} 7 & -6 & 1 \\ -5 & 6 & 1 \\ 5 & 6 & 11 \end{pmatrix} = 0 \iff 7\alpha - 6\beta + \gamma = 0, 12\beta + 12\gamma = 0$$

(where we replaced second and third equations by their sum. Picking $\beta = 1$ we get $\gamma = -1$, $\alpha = 1$, $L_3 = \begin{pmatrix} 1 & 1 & -1 \end{pmatrix}$.

So, Riemannian invariants $W_1 = u + 5w$, $W_2 = v + 6w$ and $W_3 = u + v - w$ satisfy equations $W_{1t} + 6W_{1x} = 0$, $W_{2t} + 6W_{2x} = 0$, $W_{3t} - 6W_{3x} = 0$.

(b) $W_1 = f(x - 6t)$, $W_2 = g(x - 6t)$, $W_3 = h(x + 6t)$ and from $u + 5w = f(x - 6t)$, $v + 6w = g(x - 6t)$, $u + v - w = h(x + 6t)$ we get $u = \frac{1}{12}\bigl(7f(x-6t) - 5g(x-6t) + 5h(x+6t)\bigr)$, $v = \frac{1}{2}\bigl(-f(x - 6t) + g(x - 6t) + h(x + 6t)\bigr)$, $w = \frac{1}{2}\bigl(f(x - 6t) + g(x - 6t) - h(x + 6t)\bigr)$

(c) From initial condition $f(x) = u + 5w|_{t=0} = 0$, $g(x) = v + 6w|_{t=0} = \cos x$, $h(x) = u + v - w|_{t=0} = \cos x$. Plug in (b): $u + 5w = f(x - 6t)$, $v + 6w = g(x - 6t)$, $u + v - w = h(x + 6t)$ we get $u = \frac{1}{12}\bigl(5\cos(x - 6t) + 5\cos(x + 6t)\bigr)$, $v = \frac{1}{2}\bigl(\cos(x - 6t) + \cos(x + 6t)\bigr)$, $w = \frac{1}{2}\bigl(\cos(x - 6t) - \cos(x + 6t)\bigr)$.

9) *1pt* Solve

$$\begin{cases} u_{tt} - 9u_{xx} = \sin t \cos x, \\ u|_{t=0} = 0, \quad u_t|_{t=0} = 0 \end{cases}$$

Solution. Appling d'Alambert formula

$$(\#)\ \ u(x,t) = \frac{1}{2}\bigl(u_0(x+ct) + u_0(x-ct)\bigr) + \frac{1}{2c}\int_{x-ct}^{x+ct} u_1(y)dy + \frac{1}{2c}\int_0^t d\tau \int_{x-ct+c\tau}^{x+ct-c\tau} f(y,\tau)dy$$

we get

$$u(x,t) = \frac{1}{2\cdot 3}\int_0^t d\tau \int_{x-3t+3\tau}^{x+3t-3\tau} \sin\tau \cos y dy = \frac{1}{6}\int_0^t \sin\tau\bigl(\sin(x+3t-3\tau) - \sin(x-3t+3\tau)\bigr)d\tau =$$

$$\frac{1}{12}\int_0^t \bigl(\cos(x + 3t - 4\tau) - \cos(x + 3t - 2\tau) - \cos(x - 3t + 2\tau) + \cos(x - 3t + 4\tau)\bigr)d\tau =$$

$$\frac{1}{6}\int_0^t \bigl(\cos x\cos(3t-4\tau) - \cos x\cos(3t-2\tau)\bigr)d\tau = \frac{1}{6}\cos x\bigl(-\frac{1}{4}\sin(3t-4\tau) + \frac{1}{2}\sin(3t-2\tau)\bigr)|_{\tau=0}^{\tau=t} =$$

$$\frac{1}{8}\cos x(\sin 3t - 3\sin t).$$

10) *1pt* Solve

$$(a)\begin{cases} u_{tt}-4u_{xx}=0 \quad (x>0,t>0),\\ u|_{t=0}=e^{-x}, \quad u_t|_{t=0}=0 \quad (x>0)\ ,\\ u|_{x=0}=0 \quad (t>0) \end{cases} \qquad (b)\begin{cases} u_{tt}-4u_{xx}=0 \quad (x>0,t>0),\\ u|_{t=0}=e^{-x}, \quad u_t|_{t=0}=0 \quad (x>0)\ ,\\ u_x|_{x=0}=0 \quad (t>0) \end{cases}$$

Solution. We need to continue u_0, $u_1=0$, $f=0$ to all x as odd (a) or even (b) functions: $\tilde{u}_0=\mp e^{x}$ in cases (a),(b) respectively (for $x<0$). Then (#) yields that

$$u(x,t)=\begin{cases} e^{-x}\cosh 2t & \text{for } x>2t,\\ \frac{1}{2}e^{-2t}(e^{-x}\mp e^{x}) & \text{for } x<2t. \end{cases}$$