FACULTY OF ARTS AND SCIENCE
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FINAL EXAMINATIONS, DECEMBER 2000

APM346H1F
Differential Equations

Examiner: Professor Victor Ivrii

Duration: 3 hours

AIDS: Textbooks and lecture notes. **Total: 50 points**

Problems.

FE-1 (*8 pts*) Solve by the method of characteristics

$$\begin{cases} u_{tt} - 9u_{xx} = 0, \quad x > 0,\ t > 0,\\ u|_{x=0} = t,\\ u|_{t=0} = x, \quad u_t|_{t=0} = 0. \end{cases}$$

FE-2 (*8 pts*) By a separation of variables solve the problem

$$\begin{cases} u_{tt} - 4u_{xx} = 0, \quad -\pi < x < \pi,\ t > 0,\\ u_x|_{x=-\pi} = u_x|_{x=\pi} = 0,\\ u|_{t=0} = \sin 2x, \qquad u_t|_{t=0} = 0. \end{cases}$$

FE-3 (*8 pts*) By a separation of variables solve the problem

$$\begin{cases} u_t - u_{xx} = 0, \quad -\frac{\pi}{2} < x < \frac{\pi}{2},\ t > 0,\\ u|_{x=-\frac{\pi}{2}} = u|_{x=\frac{\pi}{2}} = 0,\\ u|_{t=0} = x. \end{cases}$$

FE-4 (*8 pts*) By a method of separation find a bounded solution to the problem

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \quad r < 1,\ 0 < \theta < \frac{2\pi}{3},\\ u|_{r=1} = 1,\\ u|_{\theta=0} = u|_{\theta=\frac{2\pi}{3}} = 0 \end{cases}$$

FE-5 (*8 pts*) Find solution in the form of Fourier integral (don't calculate!):

$$\begin{cases} u_{xx} + u_{yy} = 0 & x > 0, \\ u|_{x=0} = e^{-\frac{y^2}{2}}. \end{cases}$$

FE-6 (*4+4 pts*) (a) By the method of potential (Green's function) find solution (don't calculate!) for

$$u_{xx} + u_{yy} + u_{zz} = e^{-\sqrt{x^2+y^2+z^2}}.$$

(b) Using the fact that the equation is spherically symmetric, find the spherically symmetric solution.

FE-7 (*8 pts*) Find a solution (as a spherical wave) to

$$\begin{cases} u_{tt} - u_{xx} - u_{yy} - u_{zz} = 0, \\ u|_{t=0} = 0, \\ u_t|_{t=0} = \begin{cases} 1 & \text{for } x^2 + y^2 + z^2 < 1 \\ 0 & \text{for } x^2 + y^2 + z^2 \geq 1 \end{cases} \end{cases}$$

Solutions

FE-1 From equation follows that $u(x,t) = f(x+3t)+g(x-3t)$ and initial conditions yield that $f(x)+g(x) = x$, $3f'(x)-3g'(x) = 0 \implies f(x) = g(x) = \frac{x}{2}$ and therefore as $x > 0$.

Then $f(3t) + g(-3t) = t$ as $t > 0$ and then $g(-3t) = t - f(t) = t - \frac{3}{2}t = -\frac{1}{2}t$, $g(x) = \frac{1}{6}x$ for $x < 0$. Then

$$u(x,t) = \begin{cases} x & \text{for } x > 3t > 0 \\ \dfrac{2}{3}x + t & \text{for } 3t > x > 0 \end{cases}$$

FE-2 Separating variables we get $u_n = X_n(x)T_n(t)$ with $X_n(x) = \cos\frac{n}{2}(x+\pi)$, $T_n(t) = A_n \cos nt + B_n \sin nt$, $n = 0,1,2,\dots$ and

$$u(x,t) = \frac{1}{2}(A_0 + B_0 t) + \sum_{n=1}^{\infty} \big(A_n \cos 3nt + B_n \sin 3nt\big) \sin n(x + \frac{\pi}{2})$$

and from initial conditions $B_n = 0$ $(n = 0,1,\dots)$ and

$$\frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\frac{n}{2}(x+\pi) = \sin 2x;$$

then

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos\frac{n}{2}(x+\pi)\sin 2x dx.$$

Note that $\cos\frac{n}{2}(x+\pi) = (-1)^m \cos mx$ for $n = 2m$ and integrand is an odd function and $A_n = 0$. On the other hand, $\cos\frac{n}{2}(x+\pi) = (-1)^{m+1}\sin\frac{2m+1}{2}x$ for $n = 2m+1$ and

$$A_{2m+1} = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin\frac{2m+1}{2}x \sin 2x dx = \frac{1}{\pi}\int_0^{\pi} (\cos(\frac{2m-3}{2}x - \cos\frac{2m+5}{2}x)dx =$$
$$\frac{2}{\pi}(\frac{1}{2m-3}\sin(\frac{2m-3}{2}\pi - \frac{1}{2m+5}\sin(\frac{2m+5}{2}\pi) = (-1)^m \frac{16}{\pi(2m+5)(2m-3)}$$

and

$$u(x,t) = -\sum_{m=0}^{\infty} \frac{16}{\pi(2m+5)(2m-3)} \sin(\frac{2m+1}{2}x)\cos(2m+1)t$$

FE-3 Separating variables we get $u_n(x,t) = T_n(t)X_n(x)$ with $X_n = \sin(n(x+\frac{\pi}{2})$, $T_n(t) = e^{-n^2 t}$.

Then $u(x,t) = \sum_{n=1}^{\infty} A_n e^{-n^2 t}\sin n(x+\frac{\pi}{2})$ and

$$A_n = \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin n(x + \frac{\pi}{2})dx.$$

The integrand is an odd function for odd n. For $n = 2m$ $\sin 2m(x + \frac{\pi}{2}) = (-1)^m \sin 2mx$ and

$$A_{2m} = \frac{4}{\pi}(-1)^m \int_0^{\frac{\pi}{2}} \sin 2mx x dx = -\frac{1}{m}$$

and

$$u(x,t) = \sum_{m=1}^{\infty} (-1)^m \frac{1}{m} e^{-4m^2 t} \sin 2mx.$$

FE-4 Separating variables we get $u_n(r,\theta) = R_n(r)\Theta_n(\theta)$ with $\Theta_n(\theta) = \sin \frac{3n}{2}\theta$, $n = 1, 2, \dots$ ($\Theta_n(0) = \Theta_n(\frac{2\pi}{3}) = 0$).

$R_n = r^{\frac{3n}{2}}$ then where we eliminated $r^{-\frac{3n}{2}}$ since they are unbounded at origin. So,

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n \sin \frac{3n\theta}{2}) r^{\frac{3n}{2}}$$

and $u(1,\theta) = \sum_{n=1}^{\infty} A_n \sin \frac{3n\theta}{2}) = 1$. Then

$$A_n = \frac{3}{\pi} \int_0^{\frac{2\pi}{3}} \sin \frac{3n\theta}{2} d\theta = \begin{cases} 0 & \text{for } n = 2m \\ \dfrac{4}{(2m+1)\pi}0 & \text{for } n = 2m+1 \end{cases}$$

$$u(r,\theta) = \sum_{m=0}^{\infty} \frac{4}{(2m+1)\pi} r^{\frac{3(2m+1)}{2}} \sin \frac{3(2m+1)}{2}\theta.$$

FE-5 Making Fourier transform $y \mapsto q$ we get

$$\hat{u}_{xx} - q^2 \hat{u} = 0,$$
$$\hat{u}|_{x=0} = \hat{\phi}$$

where $\phi = e^{-\frac{y^2}{2}}$ and therefore as we know $\hat{\phi} = \frac{1}{\sqrt{2\pi}} e^{-\frac{q^2}{2}}$. From equation $\hat{u} = A(q)e^{-|q|x} + B(q)e^{|q|x}$ and we should take $B(q) = 0$ to have a bounded solution. Then from initial data $A(q) = \hat{\phi} e^{-|q|t} = \frac{1}{\sqrt{2\pi}} e^{-\frac{q^2}{2} - |q|x}$ and

$$u(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{q^2}{2} - |q|x + qiy}$$

FE-6 (a) Using potential method

$$u(x,y,z) = -\frac{1}{4\pi} \iiint \left((x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2\right)^{-\frac{1}{2}} e^{-\sqrt{\xi^2+\eta^2+\zeta^2}} d\xi d\eta d\zeta.$$

(b) We are looking for spherically symmetric solution $U(r)$. Then

$$U''+\frac{2}{r}U=e^{-r}\equiv (rU)''=rU''+2U'=re^{-r}\equiv (rU)'=-(r+1)e^{-r}+\underbrace{C}_{=0}\iff$$

$$rU=(r+2)e^{-r}+\underbrace{D}_{=-2}$$

where $C=0$ to have $(rU)'$ decaying at infinity and $D=-2$ to have U finite at 0.

So, $U=e^{-r}-\frac{2}{r}(1-e^{-r})$.

FE-7 Solution must be spherically symmetric (since initial data are), and we know that then $u=\frac{1}{r}\bigl(f(t-r)-f(t+r)\bigr)$. Substituting into initial conditions, we get

$$f(-r)-f(r)=0$$

$$\frac{1}{r}\bigl(f'(-r)-f'(r)\bigr)=\begin{cases}1 & \text{for } r<1\\ 0 & \text{for } r\ge 1\end{cases}\iff -f(-r)-f(r)=\begin{cases}\dfrac{r^2}{2} & \text{for } r<1\\ \dfrac{1}{2} & \text{for } r\ge 1\end{cases}$$

Then

$$f(r)=\begin{cases}-\dfrac{r^2}{4} & \text{for } r<1\\ -\dfrac{1}{4} & \text{for } r\ge 1\end{cases}\implies \boxed{u(r,t)=\begin{cases}0 & \text{for } r\ge |t|+1\\ t & \text{for } r\ge 1\\ 0 & \text{for } r\le |t|-1\end{cases}}$$

Remark. One can find solution from general formula.