Last Name:____________________ First Name:____________________

APM 346F, GHA # 2. Oct. 7-30, 98
Front page

Read Me First! (general rules for GHA)

- Don't write your student ID anywhere
- Take care to staple your homework! (no clips, please)
- Be nice, write neatly, use pen, please!
- This GHA constitutes 10% of the final mark (1 pt = 1%)
- You must deliver it to me until 10.30 am Oct 30 in my office or earlier (on the lecture)
- Don't leave in mailboxes!

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1) *1.5 pt* Solve the problems below:

$$\begin{cases} u_{tt} - 4u_{xx} = f(x,t), \quad (t > 0, 0 < x < 1) \\ u|_{x=0} = u|_{x=1} = 0, \\ u|_{t=0} = g(x), \quad u_t|_{t=0} = h(x) \end{cases}$$

with

(a)	$f = 0,$	$g = \sin(\pi x),$	$h = \sin(2\pi x)$
(b)	$f = 0,$	$g = 1,$	$h = 0$
(c)	$f = \sin(\pi x)\sin(\pi t),$	$g = 0,$	$h = 0$
(d)	$f = \sin(\pi x)\sin(2\pi t),$	$g = 0,$	$h = 0$

2) *1.5 pt* Solve the problems below and find the limits of u as $t \to \infty$:

$$\begin{cases} u_t - 4u_{xx} = f(x,t), \quad (t > 0, 0 < x < \frac{\pi}{2}) \\ u|_{x=0} = u_x|_{x=\frac{\pi}{2}} = 0, \\ u|_{t=0} = g(x), \end{cases}$$

with

(a)	$f = 0,$	$g = \sin x,$
(b)	$f = 0,$	$g = x,$
(c)	$f = \sin x,$	$g = 0,$
(d)	$f = 1,$	$g = 0,$

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FOR MARKER'S USE ONLY	
Question	Mark
1	/1.5
2	/1.5
3	/1.5
4	/1.5
5	/2.0
6	/2.0
TOTAL	/10

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3) *1.5 pt* Solve the problems below:

$$\begin{cases} u_{xx} + u_{yy} = f(x,t), \quad (0 < x < 2\pi, 0 < y < \pi) \\ u_x|_{x=0} = u_x|_{x=2\pi} = 0, \\ u|_{y=0} = g(y), \qquad u|_{y=\pi} = h(x) \end{cases}$$

with

(a) $f = 0,\qquad g = \cos\frac{x}{2},\qquad h = \cos x$

(b) $f = 0,\qquad g = 1,\qquad h = 0$

(c) $f = \cos\frac{x}{2}\sin y,\qquad g = 0,\qquad h = 0$

(d) $f = \sin x,\qquad g = 0,\qquad h = 0$

4) *1.5 pt* Solve the problems (in polar coordinates) below:

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = f(r,\theta), \quad (1 < r < 2, 0 < \theta < 2\pi) \\ u|_{r=1} = g(\theta), \qquad u|_{r=2} = h(\theta) \end{cases}$$

with

(a) $f = 0,\qquad g = 1,\qquad h = \cos\theta$

(b) $f = 0,\qquad g = \begin{cases} 1 & 0 < \theta < \pi \\ 0 & \pi < \theta < 2\pi \end{cases},\qquad h = 0$

(c) $f = 1,\qquad g = 0,\qquad h = 0$

(d) $f = r^2\sin\theta,\qquad g = 0,\qquad h = 0$

5) *1.5 pt* Solve the problems below:

$$\begin{cases} u_{tt} - u_{xx} - u_{yy} = f(x,y,t), \quad (0 < x < \pi, 0 < y < \frac{\pi}{2}) \\ u|_{x=0} = u|_{x=\pi} = 0, \\ u|_{y=0} = u_y|_{y=\frac{\pi}{2}} = 0, \\ u|_{t=0} = g(x,y), \quad u_t|_{t=0} = h(x,y) \end{cases}$$

with

(a)	$f = 0,$	$g = 0,$	$h = \sin x \sin y$
(b)	$f = 0,$	$g = 0$	$h = 1$
(c)	$f = 1,$	$g = 0,$	$h = 0$
(d)	$f = \sin x \sin y \cos \sqrt{2}t,$	$g = 0,$	$h = 0$

6) *1.5 pt* Find equation for resonance frequences:

$$\begin{cases} u_{tt} - u_{rr} - \frac{1}{r}u_r - \frac{1}{r^2}u_{\theta\theta} = 0, \quad (1 < r < 2, 0 < \theta < 2\pi) \\ u|_{r=1} = u_r|_{r=2} = 0, \end{cases}$$

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Solutions

1 Separation of variables leads to

$$\begin{cases} X''+\lambda X=0, \\ X(0)=X(1)=0 \end{cases} \implies \lambda_n=\pi^2n^2,\ X_n(x)=\sin(\pi n x),\ n=1,2,\dots$$

and

$$T''+4\lambda T=0 \implies T_n(t)=A_n\cos(2\pi n t)+B_n\sin(2\pi n t),$$
$$u_n(x,t)=\bigl(A_n\cos(2\pi n t)+B_n\sin(2\pi n t)\bigr)\sin(\pi n x)$$

and

$$u(x,t)=\sum_{n=1}^{\infty}\bigl(A_n\cos(2\pi n t)+B_n\sin(2\pi n t)\bigr)\sin(\pi n x)$$

and for (a),(b) we need to determine coefficients from initial conditions:
(a) $A_1=1, B_2=\frac{1}{4\pi}$ and all other coefficients vanish;

$$u=\cos(2\pi t)\sin(\pi x)+\frac{1}{4\pi}\sin(4\pi t)\sin(2\pi x);$$

(b) $B_n=0$,

$$A_n=2\int_0^1 g(x)\sin(\pi n x)dx=\begin{cases}\dfrac{4}{\pi n} & n=2m+1\\ 0 & n=2m\end{cases}$$

and

$$u(x,t)=\sum_{m=0}^{\infty}\frac{4}{\pi(2m+1)}\cos\bigl(2\pi(2m+1)t\bigr)\sin\bigl(\pi(2m+1)t\bigr).$$

(c) In cases (c)-(d) ((e) is a duplicate) we need to look at

$$u(x,t)=\sum_{n=1}^{\infty}T_n(t)\sin(\pi n x)$$

and define T_n from equation and initial conditions. Obviously, $T_n=0$ unless $n=1$ and

(c) $$T_1''+4\pi^2T_1=\sin(\pi t),\ T_1(0)=T_1'(0)=0 \implies T_1=\frac{1}{3\pi^2}\sin(\pi t)-\frac{1}{6\pi^2}\sin(2\pi t)$$
$$u(x,t)=\bigl(\frac{1}{3\pi^2}\sin(\pi t)-\frac{1}{6\pi^2}\sin(2\pi t)\bigr)\sin(\pi x)$$

(d) $$T_1''+4\pi^2T_1=\sin(2\pi t),\ T_1(0)=T_1'(0)=0 \implies T_1=-\frac{t}{4\pi}t\cos(2\pi t)+\frac{1}{8\pi^2}\sin(2\pi t)$$
$$u(x,t)=\bigl(-\frac{t}{4\pi}t\cos(2\pi t)+\frac{1}{8\pi^2}\sin(2\pi t)\bigr)\sin(\pi x)$$

2 Separation of variables leads to

$$\begin{cases} X'' + \lambda X = 0, \\ X(0) = X'(\frac{\pi}{2}) = 0 \end{cases} \implies \lambda_n = (2n+1)^2 \; X_n(x) = \sin\bigl((2n+1)x\bigr),\ n = 0, 1, \ldots$$

and

$$T' + 4\lambda T = 0 \implies T_n(t) = A_n e^{-4(2n+1)^2 t}$$

$$u_n(x,t) = A_n e^{-4(2n+1)^2} \sin\bigl((2n+1)x\bigr)$$

and

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-4(2n+1)^2 t} \sin\bigl((2n+1)x\bigr)$$

and for (a),(b) we need to determine coefficients from initial condition:
(a) $A_0 = 1$ and all other coefficients vanish;

$$u = e^{-4t} \sin x;$$

(b)

$$A_n = \frac{4}{\pi} \int_0^{\frac{\pi}{2}} g(x) \sin\bigl((2n+1)x\bigr) dx = \frac{4(-1)^{n+1}}{(2n+1)^2}$$

and

$$u(x,t) = \sum_{n=0}^{\infty} \frac{4(-1)^{n+1}}{(2n+1)^2} e^{-4(2n+1)^2 t} \sin\bigl((2n+1)x\bigr);$$

in both cases $u \to 0$ as $t \to +\infty$.
(c) In cases (c)-(d) we need to look at

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin\bigl((2n+1)x\bigr)$$

and define T_n from equation and initial conditions. Obviously, $T_n = 0$ unless $n = 1$ in (c) and

$$\text{(c)} \qquad T_1' + 4T_1 = 1,\ T_1(0) = 0 \implies T_1 = \frac{1}{4}\bigl(1 - e^{-4t}\bigr), \quad u(x,t) = \frac{1}{4}\bigl(1 - e^{-4t}\bigr) \sin x$$

and $u \to v(x) = \frac{1}{3} \sin x\ t \to +\infty$ (which is the solution to $-4v'' = \sin x, v(0) = v'(\frac{\pi}{2}) = 0$).
In case (d) we decompose $f = 1$ into Fourier series:

$$1 = \sum_{n=0}^{\infty} F_n(t) \sin\bigl((2n+1)x\bigr), \quad F_n = \frac{4}{\pi} \int_0^{\frac{\pi}{2}} \sin(2n+1)x dx = \frac{4}{\pi(2n+1)}$$

and we need to solve

$$T_n' + 4(2n+1)^2 T_n = F_n, T_n(0) = 0 \implies T_n(t) = \frac{1}{\pi(2n+1)^3}\bigl(1 - e^{-4(2n+1)^2 t}\bigr) \implies$$

$$u(x,t) = \sum_{n=0}^{\infty} \frac{1}{\pi(2n+1)^3}\bigl(1 - e^{-4(2n+1)^2 t}\bigr)\bigl((2n+1)x\bigr);$$

then $u \to v(x) = \sum_{n=0}^{\infty} \frac{1}{\pi(2n+1)^3}\bigl((2n+1)x\bigr)$ as $t \to +\infty$ (and one can check that $v(x) = -\frac{x^2}{8} + \frac{\pi x}{8}$ which is the solution to $-4v'' = 1, v(0) = v'(\frac{\pi}{2}) = 0$).

3 Separation of variables leads to

$$\begin{cases} X'' + \lambda X = 0, \\ X(0) = X'(2\pi) = 0 \end{cases} \implies \lambda_n = \frac{n^2}{4}\ X_n(x) = \cos\frac{nx}{2},\ n = 0, 1, \dots$$

and

$$Y' - \lambda Y = 0 \implies Y_n(t) = A_n \cosh\frac{ny}{2} + B_n \sinh\frac{ny}{2}$$

$$u_n(x,y) = \bigl(A_n \cosh\frac{ny}{2} + B_n \sinh\frac{ny}{2}\bigr)\cos\frac{nx}{2} \quad (n = 1, 2, \dots)$$

$$Y_0 = A_0 + B_0 y, u_0(x,y) = (A_0 + B_0 y)$$

and

$$u(x,t) = \sum_{n=1}^{\infty} \bigl(A_n \cosh\frac{ny}{2} + B_n \sinh\frac{ny}{2}\bigr)\cos\frac{nx}{2}$$

and for (a),(b) we need to determine coefficients from boundary conditions:

(a) $A_1 = 1, A_1 \cosh\frac{\pi}{2} + B_1 \sinh\frac{\pi}{2} = 0 \implies A_1 = 1, B_1 = -\frac{\cosh\frac{\pi}{2}}{\sinh\frac{\pi}{2}}$, $A_2 = 0, A_2 \cosh\pi + B_2 \sinh\pi = 1 \implies B_2 = \frac{1}{\sinh\pi}$ and all other coefficients vanish;

$$u = \frac{\sinh\frac{\pi - y}{2}}{\sinh\frac{\pi}{2}}\cos\frac{x}{2} + \frac{\sinh y}{\sinh\pi}\cos x$$

(b) All the coefficients save A_0, B_0 vanish and $A_0 = 1, A_0 + B_0\pi = 0 \implies u(x,y) = \frac{\pi - y}{\pi}$.

(c) In cases (c)-(d) we need to look at

$$u(x,y) = \sum_{n=0}^{\infty} Y_n(y)\cos\frac{nx}{2}$$

and define Y_n from equation and boundary conditions. Obviously, $Y_n = 0$ unless $n = 1$ in (c) and $Y_1'' - \frac{1}{4}Y_1 = \sin y,\ Y_1(0) = Y_1(\pi) = 0$. Equation implies that $Y_1 = -\frac{4}{5}\sin y + A_1 \cos\frac{y}{2} + B_1 \sin\frac{y}{2}$ and boundary conditions yield that $A_1 = B_1 = 0$. So, $u(x,y) = -\frac{4}{5}\sin y \sin\frac{x}{2}$.

In case (d) we decompose $f = \sin x$ into Fourier series:

$$1 = \sum_{n=0}^{\infty} F_n(y)\cos\frac{nx}{2}, \quad F_n = \frac{1}{\pi}\int_0^{2\pi} \sin x \cos\frac{nx}{2}dx = \begin{cases} -\dfrac{4}{\pi(n^2-4)} & n = 2m+1 \\ 0 & n = 2m \end{cases}$$

and $Y_{2m=0}$ and and we need to solve $Y''_{2m+1} - \frac{1}{4}(2m+1)^2 Y_{2m+1} = F_n$, $Y_{2m+1}(0) = Y_{2m+1}(\pi) = 0$. Then

$$Y_{2m+1} = -\frac{4}{(2m+1)^2} + \frac{4}{(2m+1)^2\cosh\frac{(2m+1)\pi}{4}}\cosh\big((2m+1)(\frac{y}{2} - \frac{\pi}{4})\big)$$

and

$$u(x,y) = \sum_{m=0}^{\infty}\Bigg(-\frac{4}{(2m+1)^2} + \frac{4}{(2m+1)^2\cosh\frac{(2m+1)\pi}{4}}\cosh\big((2m+1)(\frac{y}{2} - \frac{\pi}{4})\big)\Bigg)\cos\frac{(2m+1)x}{2}$$

4 Separation of variables leads to

$$\begin{cases} \Theta'' + \lambda\Theta = 0, \\ \Theta(0) = \Theta(2\pi), \ '(0) = \Theta'(2\pi) \end{cases} \implies \lambda_n = n^2.\ n = 0, 1, 2, \dots;$$

$$\Theta_0 = 1; \Theta_{n,1} = \cos(n\theta), \Theta_{n,2} = \sin(n\theta)$$

and

$$r^2 R''_n + rR'_n - n^2 R_n = 0 \implies R_n = A_n r^n + B_n r^{-n} (n \ge 1), R_0 = A_0 + B_0 \log r$$

and

$$u(r,\theta) = A_0 + B_0\log r + \sum_{n=1}^{\infty}\Bigg(\big(A_n r^n + B_n r^{-n}\big)\cos(n\theta) + \big(C_n r^n + D_n r^{-n}\big)\sin(n\theta)\Bigg).$$

(a) In this case boundary conditions yield that $A_0 = 1, A_0 + B_0 \log 2 = 0,\ A_1 + B_1 = 0, 2A_1 + \frac{1}{2}B_1 = 1$ and all the other coefficients vanish.

$$u(r,\theta) = 1 - \log_2 r + \frac{2}{3}(r - r^{-1})\cos\theta.$$

(b) In this case we need to decompose $g(\theta)$ into Fourier series:

$$g(\theta) = \frac{1}{2} + \sum_{m=0}^{\infty}\frac{2}{\pi(2m+1)}\sin\big((2m+1)\theta\big)$$

and then $A_0 = \frac{1}{2}$, $A_0+B_0\log 2 = 0$, $C_{2m+1}+D_{2m+1} = \frac{2}{\pi(2m+1)}$, $C_{2m}2^{2m+1}+D_m2^{-2m-1} = 0$ and all the other coefficients vanish. So

$$u(r,\theta) = \frac{1}{2}(1-\log_2 r)+\sum_{m=0}^{\infty}\frac{2}{\pi(2m+1)\big(1-2^{4m-2}\big)}\Big(r^{-2m-1}-r^{2m+1}2^{-4m-2}\Big)\sin(2m+1)\theta$$

(c) We will look at solution in the form $u(r,\theta) = R(r)$ and then $R''+r^{-1}R' = 1$. Looking for special solution $R = ar^2$ we get $a = \frac{1}{3}$ and the general solution is $R = \frac{1}{3}r^2+A+B\log r$ and it follows from the bounday conditions that $A = -\frac{1}{3}$, $B = -\frac{1}{\log 2}$ and

$$u = \frac{1}{3}(r^2-1)-\log_2 r.$$

(d) We will look at solution in the form $u(r,\theta) = R(r)\sin\theta$ and then $R''+r^{-1}R'-R = r^2$. Looking for special solution $R = Cr^4$ we get $C = \frac{1}{15}$ and the general solution is $R = \frac{1}{15}r^4+Cr+Dr^{-1}$ and it follows from the bounday conditions that $\frac{1}{15}+C+D = 0$, $\frac{16}{15}+2C+\frac{1}{2}D = 0$ and

$$u = \left(\frac{1}{15}r^4-\frac{31}{45}r+\frac{28}{45}r^{-1}\right)\sin\theta.$$

5 Separation of variables $u(x,y,t) = X(x)Y(y)T(t)$ leads to

$$\begin{cases} X''+\mu X = 0 \\ X(0) = X(\pi) = 0 \end{cases} \Longrightarrow \mu_m = m^2, X_m = \sin mx, \quad m = 1,2,\dots$$

$$\begin{cases} Y''+\nu Y = 0 \\ Y(0) = Y'(\frac{\pi}{2}) = 0 \end{cases} \Longrightarrow \nu_n = (2n+1)^2, Y_n = \sin(2n+1)y, \quad n = 0,1,2,\dots$$

$$T''+(\mu+\nu)T = 0 \Longrightarrow T_{m,n} = A_{m,n}\cos\sqrt{m^2+(2n+1)^2}t+B_{m,n}\sin\sqrt{m^2+(2n+1)^2}t$$

(a) All the coefficients are 0 except for $m = 1, n = 2$, and

$$u(x,y,t) = \frac{1}{\sqrt{2}}\sin x\sin y\sqrt{2}t$$

(b) In this case $A_{m,n} = 0$ and

$$B_{m,n} = \frac{8}{\pi^2\sqrt{m^2+(2n+1)^2}}\int_0^{\pi}dx\int_0^{2\pi}h(x,y)\sin mx\sin(2n+1)y\,dy =$$

$$\begin{cases} \dfrac{4}{\pi^2\sqrt{m^2+(2n+1)^2}}\dfrac{2}{m}\dfrac{1}{2n+1} & m = 2p+1 \\ 0 & m = 2p \end{cases}$$

and

$$u(x,y,t)=\sum_{p=0}^{\infty}\sum_{n=0}^{\infty}\frac{16}{\pi^2(2p+1)(2n+1)\sqrt{(2p+1)^2+(2n+1)^2}}\times$$
$$\times\sin(2p+1)x\sin(2n+1)y\sin\sqrt{(2p+1)^2+(2n+1)^2}t$$

(c) In this case we decompose f into Fourier series in x, y:

$$f(x,y,t)=\sum_{p=0}^{\infty}\sum_{n=0}^{\infty}\frac{16}{\pi^2(2p+1)(2n+1)}\sin(2p+1)x\sin(2n+1)y$$

and then

$$u(x,y,t)=\sum_{p=0}^{\infty}\sum_{n=0}^{\infty}T_{p,n}(t)\sin(2p+1)x\sin(2n+1)y$$

where $T_{p,n}$ solve Cauchy problems:

$$T''+\bigl((2p+1)^2+(2n+1)^2\bigr)T=F_{p,n}(t)=\frac{16}{\pi^2(2p+1)(2n+1)},\qquad T(0)=T'(0)=0$$

and so

$$T_{p,n}=\frac{16}{\pi^2(2p+1)(2n+1)\bigl((2p+1)^2+(2n+1)^2\bigr)}\Bigl(1-\cos\sqrt{(2p+1)^2+(2n+1)^2}t\Bigr)$$

and

$$u(x,y,t)=\sum_{p=0}^{\infty}\sum_{n=0}^{\infty}\frac{16}{\pi^2(2p+1)(2n+1)\bigl((2p+1)^2+(2n+1)^2\bigr)}\times$$
$$\times\Bigl(1-\cos\sqrt{(2p+1)^2+(2n+1)^2}t\Bigr)\sin(2p+1)x\sin(2n+1)y$$

(d) In this case $u(x,y,t)=\sin x\sin yT(t)$ where $T(t)$ solves Cauchy problem

$$T''+2T=\cos\sqrt{2}t,\quad T(0)=T'(0)=0$$

and then

$$T(t)=-\frac{t}{2\sqrt{2}}\sin\sqrt{2}t$$

and

$$u(x,y,t)=-\frac{t}{2\sqrt{2}}\sin x\sin y\sin\sqrt{2}t$$

6 Separating variables we get $u(r,\theta,t)=R(r)\Theta(\theta)T(t)$ where $\Theta''+\mu\Theta=0$ and Θ is 2π-periodic and $\mu=m^2$ with $m=0,1,\ldots$ (and simple μ as $n=0$ and double μ as $n\ge 1$) and $T''+\omega^2T=0$ with characteristic frequency ω and finally

$$R''+\frac{1}{r}R'+\bigl(\omega^2-\frac{m^2}{r^2}\bigr)R=0.$$

Changing variables $z = \omega r$ we get Bessel's equation

$$z^2 \frac{d^2 R}{dz^2} + z \frac{dR}{dz} + (z^2 - m^2) R = 0$$

and it has solution of the form

$$R(r) = c_1 J_m(\omega r) + c_2 Y_m(\omega r)$$

with Bessel functions J_m, Y_m. To satisfy boundary problems we need $c_1 J_m(\omega r) + c_2 Y_m(\omega r) = 0$ as $r = 1$ and $r = 2$. This system of two equations has non-trivial solution (c_1, c_2) iff determinant is 0:

$$\begin{vmatrix} J_m(\omega) & Y_m(\omega) \\ J_m(2\omega) & Y_m(2\omega) \end{vmatrix} = J_m(\omega) Y_m(2\omega) - J_m(2\omega) Y_m(\omega) = 0$$

This is equation in question: for each $m = 0, 1, \dots$ it has an infinite serieas of roots $\omega_{m,n}$ $n = 1, 2, \dots$ which are characteristic frequences.