Nets and filters (are better than sequences)

Contents

1 Motivation

In the Section 5 of the lecture notes we saw the following two results:

Theorem 1.1. *Let $(X, \mathcal{T})$ be a Hausdorff space. Then every sequence in X converges to at most one point.*

Proposition 1.2. *Let $(X, \mathcal{T})$ be a topological space, let $A \subseteq X$, and let $a \in X$. If there is a sequence of points in A that converges to a, then $a \in \overline{A}$.*

We discussed how both of these implications really feel like they should reverse, but unfortunately neither of them do. In both cases, additionally assuming the topological space is first countable allows the implications to reverse. This is fine, but it still feels like sequences are not quite powerful enough to capture the ideas we want to capture.

The ideal solution to this problem is to define a more general object than a sequence—called a net—and talk about net convergence. That is what we will do in this note. We will also define a type of object called a filter and show that filters also furnish us with a type of convergence which turns out to be equivalent to net convergence in all ways. With these more powerful tools in place of sequence convergence, both of the above implications will reverse, among many other benefits.

We will not focus much on nets in this course (you will never be evaluated on them) but I still felt that I should mention them for the sake of completeness. We definitely will talk about filters later, so learning about them here will be helpful for you in the long run. We will also be able to fill in a big gap from first year calculus with this idea at the very end of this note.

2 More implications we wish would reverse

In addition to the two results mentioned above that do not reverse, here is another one. It is essentially a restatement of the second result above in terms that are perhaps easier to digest.

Proposition 2.1. *Let $(X, \mathcal{T})$ be a topological space, let $C \subseteq X$ be closed, and let $x \in X$. If there is a sequence of points in C that converges to x, then $x \in C$.*

Proof. **Exercise.** (This is essentially the same as the second result mentioned in the previous section.) □

This is another implication we wish would reverse. That is, it feels as though a set that contains all the limit points of all of its sequences should be closed. Again, this is unfortunately not the case (try to come up with a counterexample!).

This property has a name though:

Definition 2.2. *Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. A is said to be sequentially closed if whenever a sequence $\{x_n\}_{n=1}^{\infty}$ of elements of A converges to a point x, then $x \in A$.*

In this language, the proposition above says that every closed subset of a topological space is sequentially closed, though the converse is not true.

Next, we give another example of an implication we wish would reverse. This requires another new definition beforehand.

Definition 2.3. *Let $(X, \mathcal{T})$ be an infinite topological space, and let $A \subseteq X$ be infinite. A point $x \in X$ is called an accumulation point of A (more properly an ω-accumulation point of A) if every open set containing x contains infinitely many points of A.*

So the property of being an accumulation point of A looks strictly stronger than the property of being in the closure of A. Instead of any open set around x simply intersecting A, we require this intersection to be infinite. For our purposes, we will restrict our attention to accumulation points of sequences. To be clear, given a sequence $\{x_n\}$, an accumulation point of the sequence is a point x such that any open set containing x contains x_n for infinitely many n (but not necessarily a tail of the sequence).

This all seems fine, so where is the problem? What implication does not reverse here?

Proposition 2.4. *Let $(X, \mathcal{T})$ be a topological space and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X (not necessarily convergent). Let $\{x_{n_k}\}_{k=1}^{\infty}$ be a subsequence of our sequence that converges to a point x. Then x is an accumulation point of the original sequence.*

Proof. Let $\{x_{n_k}\}_{k=1}^{\infty}$ be a subsequence as in the proposition, converging to x. Let U be an open set containing x. Then by definition of sequence convergence there is an $N \in \mathbb{N}$ such that for all $k \geq N$, $x_{n_k} \in U$. Then the points $x_{n_N}, x_{n_{N+1}}, x_{n_{N+2}}, x_{n_{N+3}}, \ldots$ are all in U, and so x is an accumulation point of the sequence. □

This proposition is another implication we wish would reverse. It feels as though every accumulation point should be the limit of a subsequence. Again though, this implication fails. (Try to come up with a counterexample!) And again, if you additionally assume the space is first countable, the implication *does* reverse.

By now I hope you have the feeling that first countability is a property designed to characterize the fact that sequence convergence determines what is going on in a topology. Getting a little more abstract, we have seen examples of very different topologies in which all the same sequences converge (though we have not yet remarked on it).

Example 2.5. Consider the two spaces $(\mathbb{R}, \mathcal{T}_{\text{co-countable}})$ and $(\mathbb{R}, \mathcal{T}_{\text{discrete}})$. Then a sequence of real numbers converges in one of these spaces if and only if it converges in the other. Show this!

So even though these are two very different-feeling topologies on the real numbers, it is impossible to detect any difference between them just be analyzing convergent sequences. This is good evidence that sequence convergence is not a powerful enough concept to encapsulate all the information a topology can carry, and that first countability is a property designed to patch this hole. The idea with net and filter convergence is to design a more powerful notion of convergence that obviates the need for such a patch.

3 Nets

Recall that a sequence in a set X is simply a function $x : \mathbb{N} \to X$. In order to generalize this idea, we are going to allow ourselves to consider functions whose domains are a more general sort of object, called a directed set.

Definition 3.1. *Let $\mathcal{D}$ be a set, and let $\leq$ be a relation on $\mathcal{D}$ satisfying the following properties:*

1. *$\leq$ is reflexive: for any $x \in \mathcal{D}$, $x \leq x$.*
2. *$\leq$ is transitive: for any $a, b, c \in D$, if $a \leq b$ and $b \leq c$, then $a \leq c$.*
3. *$\leq$ is directed: for any $a, b \in \mathcal{D}$ there exists an element $c \in \mathcal{D}$ such that $a \leq c$ and $b \leq c$.*

   A pair $(\mathcal{D}, \leq)$ satisfying these three properties is called a directed set.

Example 3.2. Some examples of simple directed sets:

1. $\mathcal{D} = \mathbb{N}$ with its usual ordering relation $\leq$.
2. $\mathcal{D} = \{\, \{n, n+1, n+2, \ldots\} \subseteq \mathbb{N} : n \in \mathbb{N} \,\}$, with the subset relation $\subseteq$ or the superset relation $\supseteq$. The latter is more useful, as we will soon see.
3. Let $(X, \mathcal{T})$ be a topological space, and let $x \in X$. Then the set

   $$\mathcal{D}_x := \{\, U \in \mathcal{T} : x \in U \,\}$$

   is a directed set when equipped with the either the subset relation $\subseteq$, or more usefully the superset relation $\supseteq$.
4. If $(\mathcal{D}_1, \leq_1)$ and $(\mathcal{D}_2, \leq_2)$ are directed sets, then $(\mathcal{D}_1 \times \mathcal{D}_2, \leq)$ is a directed set where $\leq$ is defined by

   $$(a, b) \leq (x, y) \text{ if and only if } a \leq_1 x \text{ and } b \leq_2 y.$$

   (Show this!)

Definition 3.3. *A net in a set X is a map $w : \mathcal{D} \to X$, where $\mathcal{D}$ is a directed set.*

Remark 3.4. Note immediately that a sequence is a net, since $(\mathbb{N}, \leq)$ is a directed set.

Definition 3.5. *If $(X, \mathcal{T})$ is a topological space and $w : \mathcal{D} \to X$ is a net, we say that w converges to a point $x \in X$ if for any open set U containing x, there is a $d \in \mathcal{D}$ such that $T_d := \{\, w(e) : d \leq e \in \mathcal{D} \,\} \subseteq U$. We call a set of the form T_d a tail of the net.*

If a net w converges to x, we denote this simply by $w \to x$, and refer to x a limit point of the net.

Take a moment to verify to yourself that the use of the word "tail" in this context agrees with its use in the context of sequences, and that in the case where $\mathcal{D} = \mathbb{N}$ with its usual ordering, this agrees with the usual definition of sequence convergence.

Just to give some more language to use, we usually say that a net $w : \mathcal{D} \to X$ is "eventually in A" if A contains some tail of w. In this language, we say that $w \to x$ if w is eventually in any open set containing x.

As promised, let's use this to prove a theorem we wish was true about sequences.

Theorem 3.6. *Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. Then $x \in \overline{A}$ if and only if there is a net $w : \mathcal{D} \to A$ such that $w \to x$.*

Proof. ($\Leftarrow$). This proof is essentially the same as for sequences.

Suppose $w : \mathcal{D} \to A$ is a net such that $w \to x$. We want to show $x \in \overline{A}$. So fix an open set U containing x. By definition of net convergence, there is a $d \in \mathcal{D}$ such that $w(e) \in U$ for all $e \geq d$. Since $w(e) \in A$ for all $e \in \mathcal{D}$, in particular the intersection $U \cap A$ is nonempty.

($\Rightarrow$). Here is where nets help you.

Suppose $x \in \overline{A}$. We need to show that there is a net w that converges to x. To do this we first need to define a directed set to be the domain of our net.

Let $\mathcal{D}_x = \{ U \in \mathcal{T} : x \in U \}$. Equipped with the superset relation, this is a directed set. To be clear, we are saying $U \leq V$ if and only if $U \supseteq V$; going up in the order on $\mathcal{D}_x$ is like "honing in on x".

By the definition of $\overline{A}$, for every $U \in \mathcal{D}_x$ we can fix a point $x_U \in U \cap A$. Define $w : \mathcal{D}_x \to A$ by $w(U) = x_U$. Then w is a net, and we claim that $w \to x$.

Indeed, fix an open set U containing x. Then $U \in \mathcal{D}_x$, and so for all $V \geq U$ in $\mathcal{D}_x$ (ie. for all $V \subseteq U$) we have:

$$w(V) = x_V \in V \cap A \subseteq U \cap A \subseteq U.$$

This shows that the tail $T_U = \{ V \in \mathcal{D}_x : U \leq V \} = \{ V \in \mathcal{D}_x : V \subseteq U \}$ of the net is contained in U, as required. □

Notice that this proof worked because the net we defined "knows about" *all* of the open sets containing x. When a space is first countable, the collection of all open sets containing x is somehow described by a countable subcollection, which lends itself to defining sequences. Nets are not bound by this restriction though, and in fact we can use the directed set structure of this collection of open sets itself to index an object that is richer than a sequence.

Corollary 3.7. *A subset A of a topological space $(X, \mathcal{T})$ is closed if and only if the limit points of all convergent nets in A are again in A.*

Proof. By the previous result, we know that the closure $\overline{A}$ of a set A is precisely the set of all limit points of nets in A. The result then follows from the fact that A is closed if and only if $A = \overline{A}$. □

So if we give a name to the analogue of "sequentially closed" for nets—call a set A netly closed if the limit point of every convergent net in A is in A—we have shown that netly closed is just the same as closed.

Finally, we prove nets reverse the implication in Theorem 1.1.

Theorem 3.8. *Let $(X, \mathcal{T})$ be a topological space. Then the space is Hausdorff if and only if every net in X converges to at most one point.*

Proof. ($\Rightarrow$). This proof is essentially the same as the proof for sequences.

Suppose $(X, \mathcal{T})$ is Hausdorff. Let $w : \mathcal{D} \to X$ be a net that converges to a point x, and suppose that $y \neq x$. We want to show that $w \not\to y$. Choose disjoint open sets U and V containing x and y, respectively. By definition of net convergence, there is some tail T_a of the net in U, and therefore this tail is disjoint from V. This means the net cannot converge to y, since if some other tail T_b was in V, we could use directedness to find a $c \in \mathcal{D}$ such that $a \leq c$ and $b \leq c$, from which it would follow that the tail T_c would have to be in both U and V, which is impossible.

($\Leftarrow$). We prove this by contrapositive. Suppose $(X, \mathcal{T})$ is not Hausdorff, and we will show that there is a net converging to more than one point.

Find two points $x, y \in X$ such that any pair of open sets U and V containing x and y, respectively, intersect one another. Again using the idea of the directed set $\mathcal{D}_x = \{ U \in \mathcal{T} : x \in U \}$ with the superset relation, define the directed set $\mathcal{D} = \mathcal{D}_x \times \mathcal{D}_y$, with

$$(U_1, V_1) \leq (U_2, V_2) \text{ if and only if } U_1 \supseteq U_2 \text{ and } V_1 \supseteq V_2.$$

For each $U \in \mathcal{D}_x$ and $V \in \mathcal{D}_y$, fix a point $x_{U,V} \in U \cap V$. Then the function $w : \mathcal{D} \to X$ given by $w(U, V) = x_{U,V}$ is a net that converges to both x and y. (Prove this!) □

Again, the ($\Leftarrow$) direction of this proof (the one which is not true for sequences) worked because the net we created "knew about" all of the open sets containing x and all the open sets containing y.

4 Subnets

We can also generalize the definition of an accumulation point of a sequence in a natural way:

Definition 4.1. *Let $(X, \mathcal{T})$ be a topological space, let $w : \mathcal{D} \to X$ be a net, and fix $x \in X$. Then x is called an accumulation point of w if for every open set U containing x and every $d \in \mathcal{D}$, there is $e \in \mathcal{D}$ with $d \leq e$ such that $w(e) \in U$.*

It may not be obvious that this generalizes the definition of accumulation point of a sequence, but it does. To see this, note that in Definition 2.3 we could equivalently have said that x is an accumulation point of $\{x_n\}$ if given any open set U containing x and $m \in \mathbb{N}$, there is an $n \geq m$

such that $x_n \in U$. Essentially this says that the set of indices n for which $x_n \in U$ is unbounded, or in other words infinite.

The definition of an accumulation point of a net just above exactly generalizes this to what might be called an "unbounded" subset of $\mathcal{D}$—one that has elements above any given element of $\mathcal{D}$. The usual name for a subset of a directed set that has this property is a cofinal set. Just to formalize this:

Definition 4.2. *A subset $\mathcal{D}'$ of a directed set $\mathcal{D}$ is called cofinal if for every $d \in \mathcal{D}$ there exists an element $e \in \mathcal{D}'$ such that $d \leq e$.*

Exercise 4.3. Show that if $(\mathcal{D}, \leq)$ is a directed set and $\mathcal{D}' \subseteq \mathcal{D}$ is cofinal, then $(\mathcal{D}', \leq')$ is a directed set, where $\leq'$ is the restriction of $\leq$ to $\mathcal{D}'$.

In sequence world, we know that a subsequence of $\{x_n\}_{n=1}^{\infty}$ is a sequence of the form $\{x_{n_k}\}_{k=1}^{\infty}$. Stated in more generalizable terms, given a sequence $\{x_n\}_{n \in \mathbb{N}}$, a subsequence is a sequence of the form $\{x_{h(k)}\}_{k \in \mathbb{N}}$ where h is an increasing, unbounded function $h : \mathbb{N} \to \mathbb{N}$ (ie. an increasing function whose range is cofinal in $\mathbb{N}$). To translate between this and the usual way we talk about subsequences, use the function $h(k) = n_k$.

This way of talking about subsequences is what we need to define subnets. This definition is only intuitive (and it is intuitive) once you understand the more general way of defining subsequences just mentioned.

Definition 4.4. *Let $w : \mathcal{D} \to X$ and $v : \mathcal{E} \to X$ be nets (where $(\mathcal{D}, \leq)$ and $(\mathcal{E}, \preceq)$ are both directed sets, with no relationship assumed between them). We say that v is a subnet of w if there is a function $h : \mathcal{E} \to \mathcal{D}$ such that:*

1. *h is monotone: if $\alpha \preceq \beta$, then $h(\alpha) \leq h(\beta)$.*
2. *h is cofinal, meaning $h(\mathcal{E})$ is a cofinal subset of $\mathcal{D}$.*
3. *$v(\alpha) = w(h(\alpha))$ for all $\alpha \in \mathcal{E}$.*

Exercise 4.5. Given a sequence $\{x_n\}_{n=1}^{\infty}$ and a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ in the usual sense, convince yourself that the subsequence is a subnet of the sequence.

With sequences, we know that if $x_n \to x$, then any subsequence will also converge to x. The same is true in this context:

Proposition 4.6. *Let $w : \mathcal{D} \to X$ be a convergent net, and assume $w \to x$. If w' is a subnet of w, then $w' \to x$ also.*

So this notion of subnet feels right in at least this sense. Let's immediately show that nets allow us to reverse the implication in Proposition 2.4.

Theorem 4.7. *Let $(X, \mathcal{T})$ be a topological space, $\mathcal{D}$ a directed set and $w : \mathcal{D} \to X$ a net. Fix $x \in X$. Then x is an accumulation point of w if and only if there is a subnet w' of w such that $w' \to x$.*

Proof. For clarity's sake, we use $\leq$ and $\preceq$ to denote the orderings on $\mathcal{D}$ and $\mathcal{E}$, respectively, and we use Greek letters for elements of $\mathcal{E}$.

($\Leftarrow$). Let $w : \mathcal{D} \to X$ be a net, let $w' : \mathcal{E} \to X$ be a subnet of w, and assume $w' \to x$. We want to show x is an accumulation point of w.

So fix an arbitrary open set U containing x, and $d \in \mathcal{D}$. We want to show that there is an element $e \geq d$ such that $w(e) \in U$. Let h be the function witnessing that w' is a subnet of w. Since h is cofinal, there is an $\alpha \in \mathcal{E}$ such that $h(\alpha) \geq d$. Also since $w' \to x$, there is a $\beta \in \mathcal{E}$ such that the tail

$$T_\beta = \left\{ w'(\gamma) \,:\, \beta \preceq \gamma \in \mathcal{E} \right\} \subseteq U.$$

Fix any γ such that $w'(\gamma) \in T_\beta$. Since $\mathcal{E}$ is directed, we can find an element $\delta \in \mathcal{E}$ such that $\alpha \preceq \delta$ and $\gamma \preceq \delta$. Define $e = h(\delta) \in \mathcal{D}$. Then by the monotonicity of h, $e = h(\delta) \geq h(\alpha) \geq d$, and $w(e) = w(h(\delta)) = w'(\delta) \in U$, as required.

($\Rightarrow$). Suppose x is an accumulation point of w. We want to find a subnet $w' : \mathcal{E} \to X$ of w that converges to x. To do this we need to define a directed set $\mathcal{E}$ and a map $h : \mathcal{E} \to \mathcal{D}$ witnessing that w' is a subnet of w.

Let $\mathcal{D}_x = \{ U \in \mathcal{T} \,:\, x \in U \}$ be the collection of open subsets of X that contain x. As we saw in Example 3.2, $(\mathcal{D}_x, \supseteq)$ is a directed set. Define the set:

$$\mathcal{E} := \{ (d, U) \in \mathcal{D} \times \mathcal{D}_x \,:\, w(d) \in U \}.$$

Give this set the ordering induced from the ordering on $\mathcal{D} \times \mathcal{D}_x$, described in Example 3.2.4 where $\mathcal{D}$ has the ordering it came with and $\mathcal{D}_x$ is ordered by reverse inclusion. That is, say that

$$(d, U) \leq_0 (e, V) \text{ if and only if } d \leq e \text{ and } U \supseteq V.$$

(the subscript 0 on the order relation is just there to distinguish it from the order on $\mathcal{D}$).

It is easy to see that this relation makes $(\mathcal{E}, \leq_0)$ a directed set. Transitivity and reflexivity follow from those properties on $\mathcal{D}$ and $\mathcal{D}_x$. For directedness, suppose we have (d_1, U_1) and (d_2, U_2). Let $V = U_1 \cap U_2$, and note that $x \in V$. By the directedness of $\mathcal{D}$ there is an $e \in \mathcal{D}$ such that $d_1 \leq e$ and $d_2 \leq e$. Since x is an accumulation point of w, there is an $e' \geq e$ such that $w(e') \in V$. Therefore, $(e', V) \in \mathcal{E}$, and $(d_1, U_1) \leq_0 (e', V)$ and $(d_2, U_2) \leq_0 (e', V)$, as required.

So $\mathcal{E}$ is a directed set. Define $h : \mathcal{E} \to \mathcal{D}$ by $h(d, U) = d$. h is clearly monotone, and it is also cofinal; it is actually surjective, since for any $d \in \mathcal{D}$, $h(d, X) = d$. Use this function to define the subnet $w' : \mathcal{E} \to X$ by $w'(d, U) := w(h(d, U)) = w(d) \in X$.

It remains to show that $w' \to x$. To this end, let U be an open set containing x. Since x is an accumulation point of w, we can in particular find a $d \in \mathcal{D}$ such that $w(d) \in U$. But then $(d, U) \in \mathcal{E}$, and if (e, V) is an element of $\mathcal{E}$ such that $(d, U) \leq_0 (e, V)$, then in particular $U \supseteq V$, so $w'(e, V) = w(e) \in V \subseteq U$. This shows that the tail

$$T_{(d,U)} = \left\{ w'(e, V) \,:\, (d, U) \leq_0 (e, V) \right\} \subseteq U,$$

And therefore that $w' \to x$. ∎

What a mess. I know this proof seems daunting, but it really is almost entirely devoid of ideas. It's mostly just unwinding definitions. The one piece of cleverness was in defining $\mathcal{E}$ in the second part.

So, I hope the preceding two sections have convinced you that nets are better than sequences for the purposes of topology. In fact, the convergence of nets completely characterizes a topology (which we realized was not true of sequences when we saw the examples of $(\mathbb{R}, \mathcal{T}_{\text{discrete}})$ and $(\mathbb{R}, \mathcal{T}_{\text{co-finite}})$ which were very different topologies with exactly the same convergent sequences). This is worth stating as a theorem, which I encourage you to try to prove. It is not too hard given everything we have done so far.

Theorem 4.8. *Let X be a set and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on X. Then the following are equivalent*

1. $\mathcal{T}_1 = \mathcal{T}_2$.
2. *For every net $w : \mathcal{D} \to X$, w converges in $\mathcal{T}_1$ if and only if it converges in $\mathcal{T}_2$.*

5 Filters

Filters are a type of object *a priori* unrelated to nets, but it turns out they are intimately connected. While students in this class are not required to know about nets, we will deal with at least one major proof and at least one section towards the end of the course that uses filters heavily.

Without delay, let's define them. We give the definition of a filter on a set (not necessarily a topological space) because it is of independent interest.

Definition 5.1. *Let X be a set. A nonempty collection $\mathcal{F} \subseteq \mathcal{P}(X)$ is called a filter on X if the following three properties are satisfied:*

1. $\emptyset \notin \mathcal{F}$.
2. *$\mathcal{F}$ is closed upwards: if $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$.*
3. *$\mathcal{F}$ is closed under finite intersections: if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.*

A filter $\mathcal{F}$ on a set X is called an ultrafilter if it is not properly contained in any other filter on X. A subset $\mathcal{F}'$ of a filter $\mathcal{F}$ that is itself a filter is called a subfilter of $\mathcal{F}$.

Example 5.2.

1. Given a nonempty set X, $\mathcal{F} = \{X\}$ is a filter on X. It is a trivial example that never comes up, really.

2. If $(X, \mathcal{T})$ is a topological space and $x \in X$, the collection

$$\mathcal{F}_x = \{\, A \subseteq X \;:\; \exists U \in \mathcal{T} \text{ such that } x \in U \subseteq A \,\}$$

is a filter on X, called the neighbourhood filter of x. (Note that this is the same object we were talking about earlier in the context of directed sets and nets, for example in the proof of Theorem 3.6.)

3. More generally, the collection $\mathcal{U}_x = \{ A \subseteq X : x \in A \}$ is a filter on X, and in fact an ultrafilter on X. We can see this because if A is any subset of X not in the filter, then $x \notin A$. But $\{x\}$ is in $\mathcal{U}_x$. So if $\mathcal{F}$ is a filter that properly contains $\mathcal{U}_x$ and $A \in \mathcal{F}$, the third property of a filter would mandate that $A \cap \{x\} = \emptyset \in \mathcal{F}$, which cannot happen. Ultrafilters of this form (all the sets that contain x for some fixed x) are called *principle ultrafilters*, and are relatively boring objects. Still, they will be important for us later in the course.

4. The collection $\mathcal{F} = \{ \{n, n+1, n+2, n+3, \dots\} : n \in \mathbb{N} \}$ of tails of the natural numbers is a filter on $\mathbb{N}$.

5. More generally, if X is an infinite set, the collection $\mathcal{F} = \{ A \subseteq X : X \setminus A \text{ is finite} \}$ of co-finite subsets of X is a filter, usually called the *Fréchet filter*. An important property of the Frèchet filter on an infinite set is that any other filter containing it (in particular any ultrafilter containing it) cannot contain any finite sets. (Check this!)

The third property in the definition above is worth generalizing.

Definition 5.3. *A collection $\mathcal{S} \subseteq \mathcal{P}(X)$ of subsets of a set X is said to have the finite intersection property if any intersection of finitely many members of $\mathcal{S}$ is nonempty. ie. for every $A_1, A_2, \dots, A_n \in \mathcal{S}$,*

$$\bigcap_{k=1}^{n} A_k \neq \emptyset.$$

It turns out that collections having this property actually generate filters in a natural way. They act sort of like subbases of topologies:

Proposition 5.4. *Any collection $\mathcal{S} \subseteq \mathcal{P}(X)$ with the finite intersection property generates a unique, smallest filter that contains it.*

Proof. **Exercise.** (Hint: First add finite intersections, then add supersets. Prove the resulting collection is a filter.) □

Example 5.5. If $(X, \mathcal{T})$ is a topological space and $x \in X$, the collection $\{ U \in \mathcal{T} : x \in U \}$ has the finite intersection property (it's actually closed under finite intersections, by definition of a topology). The filter it generates is the neighbourhood filter introduced in Example 5.2.2.

You should always think of a filter $\mathcal{F}$ on a set X as a definition of "largeness". That is, a subset $A \subseteq X$ is "large according to $\mathcal{F}$" if $A \in \mathcal{F}$. Ultrafilters in particular are good at this due to the following property, which says that according to an ultrafilter, either a set A or its complement must be a large.

Proposition 5.6. *Let X be a set and $\mathcal{U}$ a filter on X. Then $\mathcal{U}$ is an ultrafilter if and only if for any subset $A \subseteq X$, either $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$.*

Proof. ($\Rightarrow$) Let $\mathcal{U}$ be an ultrafilter on X, and suppose A is a nonempty subset of X such that $A \notin \mathcal{U}$. We want to show that $X \setminus A \in \mathcal{U}$.

By the maximality of $\mathcal{U}$, it must be the case that $\mathcal{U} \cup \{A\}$ is not a filter, and moreover that it does not have the finite intersection property (if it did, it would generate a filter that contains $\mathcal{U}$, again contradicting maximality). That means there is a $B \in \mathcal{U}$ such that $A \cap B = \emptyset$. But then $B \subseteq X \setminus A$, and therefore $X \setminus A \in \mathcal{U}$ since $\mathcal{U}$ is closed upwards.

($\Leftarrow$) **Exercise.** (Hint: Suppose $\mathcal{U}$ is properly contained in another filter, so in particular there is some set A such that $\mathcal{U} \sqcup \{A\}$ is contained in this larger filter. Use this set to contradict the property you assumed.) □

Even better, ultrafilters think any subset of a large set is either large, or its complement *in the large set* is large.

Corollary 5.7. *Let $\mathcal{U}$ be an ultrafilter on X, and let $A \in \mathcal{U}$. Given a subset $B \subseteq A$, either $B \in \mathcal{U}$ or $A \setminus B \in \mathcal{U}$.*

Proof. **Exercise.** □

Anyway, the reason we are defining filters in this note is because they provide us with another notion of convergence. There is more than one way to define this, and for the moment we will use the simplest one to state and then prove the connection with nets. Recall from the examples above that the collection of all open sets containing a point x generates a filter $\mathcal{F}_x$ called the neighbourhood filter of x.

Definition 5.8. *Let $(X, \mathcal{T})$ be a topological space, $\mathcal{F} \subseteq \mathcal{P}(X)$ a filter on X, and $x \in X$. Then $\mathcal{F}$ is said to converge to x if $\mathcal{F}_x \subseteq \mathcal{F}$. In this case we write $\mathcal{F} \to x$.*

The more intuitive way to think about this definition is that $\mathcal{F}$ converges to x if every open set containing x is an element of $\mathcal{F}$. This is equivalent to saying $\mathcal{F}_x \subseteq \mathcal{F}$ though, since $\mathcal{F}$ is closed upwards.

Now the definition of an accumulation point in this context.

Definition 5.9. *Let $\mathcal{F}$ be a filter on a topological space $(X, \mathcal{T})$, and let $x \in X$. Then x is called an accumulation point of $\mathcal{F}$ if for every $F \in \mathcal{F}$ and every open set U containing x, $F \cap U \neq \emptyset$.*

Equivalently, x is an accumulation point of $\mathcal{F}$ if $x \in \overline{F}$ for every $F \in \mathcal{F}$.

One seeming disadvantage of filter convergence is that not much interesting can be said about convergence of subfilters. If $\mathcal{F}$ is a filter, $\mathcal{F}' \subseteq \mathcal{F}$ is a subfilter and $\mathcal{F}' \to x$, then $\mathcal{F} \to x$. So there is no hope of a result of the form: x is an accumulation point of $\mathcal{F}$ if and only if some subfilter of $\mathcal{F}$ converges to x. However with ultafilters we can still get a satisfying result.

Proposition 5.10. *Let $(X, \mathcal{T})$ be a topological space. If $\mathcal{F}$ is a filter on X and $\mathcal{F} \to x$, then x is an accumulation point of $\mathcal{F}$. Conversely, if x is an accumulation point of an ultrafilter $\mathcal{U}$ on X, then $\mathcal{U} \to x$.*

Proof. First, suppose $\mathcal{F}$ is a filter and $\mathcal{F} \to x$. Then $\mathcal{F}_x \subseteq \mathcal{F}$, and so in particular for any open set $U \in \mathcal{F}_x$ and any $F \in \mathcal{F}$, $U \cap F \neq \emptyset$ since $\mathcal{F}$ is closed under finite intersections and $\emptyset \notin \mathcal{F}$.

Second, suppose x is an accumulation point of an ultrafilter $\mathcal{U}$ on X. We want to show that $\mathcal{F}_x \subseteq \mathcal{U}$, so fix $F \in \mathcal{F}_x$. Since x is an accumulation point of $\mathcal{U}$ and F contains an open set containing x, we have that $U \cap F \neq \emptyset$ for every $U \in \mathcal{U}$. But then the collection $\mathcal{U} \cup \{F\}$ has the finite intersection property, and so it must be contained in some filter. That filter must be $\mathcal{U}$ itself, since $\mathcal{U}$ is not properly contained in any other filter on X. □

A nontrivial fact about filters (the proof requires Zorn's Lemma) is that every filter $\mathcal{F}$ can be extended to an ultrafilter. That leads us to the following restatement of the previous proposition, which is more in line with how we stated the result about accumulation points of nets.

Proposition 5.11. *Let $(X, \mathcal{T})$ be a topological space. Then x is accumulation point of a filter $\mathcal{F}$ if and only if there exists a filter $\mathcal{G} \supseteq \mathcal{F}$ such that $\mathcal{G} \to x$.*

6 The connection between nets and filters

We have now defined two very different looking notions of convergence. In this section we will show that they are equivalent in a very strong way. We will first define a concrete way of producing a filter from every net, and a way of producing nets from every filter. Then we will show that the convergence properties of one carry over to the other and vice versa.

Definition 6.1. *Let $\mathcal{F}$ be a filter on X. Define an ordering $\leq$ on $\mathcal{F}$ by $F_1 \leq F_2$ if and only if $F_1 \supseteq F_2$. (Going up in the order corresponds to getting smaller as a set, so as we go up in the order we hone in on points in X) Then $(\mathcal{F}, \leq)$ is a directed set.*

Any net $w : \mathcal{F} \to X$ whose domain is this directed set and which has the property that $w(F) \in F$ for every $F \in \mathcal{F}$ is called a derived net of $\mathcal{F}$.

(Note that derived nets are not unique. Any net using the directed set we defined as its domain and having the property that $w(F) \in F$ is a derived net of $\mathcal{F}$).

Recall that if $A \subseteq X$ we say a net $w : \mathcal{D} \to X$ is "eventually in A" if A contains some tail $T_d = \{\, w(e) \,:\, d \leq e \in \mathcal{D} \,\}$ of the net.

Definition 6.2. *Let $\mathcal{D}$ be a directed set, and let $w : \mathcal{D} \to X$ be a net on X. Define:*

$$\mathcal{F}_w = \{\, F \subseteq X \,:\, w \text{ is eventually in } F \,\}.$$

Then $\mathcal{F}_w$ is called the derived filter of w.

Proof that $\mathcal{F}_w$ is a filter. Clearly $\emptyset \notin \mathcal{F}_w$. It is also clear that $\mathcal{F}_w$ is closed upwards: if w is eventually in F and $F \subseteq F'$, then w is eventually in F'.

It remains only to show that $\mathcal{F}_w$ is closed under finite intersections. So let $F_1, F_2 \in \mathcal{F}$. By definition of $\mathcal{F}_w$ there are tails T_{d_1} and T_{d_2} of w in F_1 and F_2, respectively. Using the fact that $\mathcal{D}$ is directed, let $e \in \mathcal{D}$ be such that $d_1 \leq e$ and $d_2 \leq e$. Then $T_e \subseteq F_1 \cap F_2$, and so $F_1 \cap F_2 \in \mathcal{F}_w$. □

Okay, so we can get filters from nets and we can get nets from filters. The point of this is the following:

Theorem 6.3. *Let $(X, \mathcal{T})$ be a topological space, and let $x \in X$.*

1. *If $w : \mathcal{D} \to X$ is a net, then $w \to x$ if and only if the derived filter $\mathcal{F}_w \to x$.*
2. *If $\mathcal{F}$ is a filter on X, then $\mathcal{F} \to x$ if and only if every derived net of $\mathcal{F}$ converges to x.*

Proof.

1. **Exercise.** (There's almost nothing to do here. It's immediate from the definitions.)

2. $(\Rightarrow)$. Assume $\mathcal{F} \to x$, and let $w : \mathcal{F} \to X$ be any derived net of $\mathcal{F}$. Fix an open set U containing x. We want to show that a tail of the net is in U.

   By assumption, $U \in \mathcal{F}$ ($\mathcal{F} \to x$ implies that $\mathcal{F}$ contains every open set containing x). We will show that the tail $T_U \subseteq U$. Indeed, given $V \in \mathcal{F}$ with $U \leq V$ (ie. $V \subseteq U$), we immediately have $w(V) \in V \subseteq U$. So $T_U \subseteq U$, and therefore $w \to x$.

   $(\Leftarrow)$. Assume every derived net $w : \mathcal{F} \to X$ converges to x, and assume for a contradiction that $\mathcal{F} \not\to x$. This means that there is some $A \subseteq X$ containing an open set U containing x such that $A \notin \mathcal{F}$. In particular this means that $U \notin \mathcal{F}$ or in other words $F \neq U$ for all $F \in \mathcal{F}$. Let w be any derived net of $\mathcal{F}$ such that $w(V) \in V \setminus U$ for all $V \in \mathcal{F}$. Then clearly this derived net does not converge to x (no point in $\mathcal{F}$ gets mapped into U by w, let alone a tail of w), contradicting the assumption that all derived nets of $\mathcal{F}$ do converge to x.

□

The connection extends to subnets and superfilters. We already saw with Proposition 5.11 that bigger filters (superfilters) relate to filters in a similar way as subnets and subsequences relate to nets and sequences. This proposition, whose proof I will omit because it is quite long and not particularly interesting, reinforces that:

Proposition 6.4. *Let $(X, \mathcal{T})$ be a topological space, suppose $\mathcal{F}$ is a filter on X and moreover that $\mathcal{F} = \mathcal{F}_w$ for some net $w : \mathcal{D} \to X$. (This last assumption does not cause a loss of generality, as every filter can be expressed as the derived filter of some net. Try to prove that!)*

Then:

- *If w' is a subnet of w, then $\mathcal{F}_{w'} \supseteq \mathcal{F}_w$.*
- *If $\mathcal{G}$ is a filter that contains $\mathcal{F}$, then $\mathcal{G} = \mathcal{F}_{w'}$ for some subnet w' of w.*

All of this is meant to convince you that the relationship between nets and filters is deep and interesting. Most things that can be stated in terms of nets can be stated in terms of filters and vice versa, and most of the time the results carry through without much difficulty.

7 The payoff

We did it! The two notions of convergence are equivalent in a very strong sense. The following theorems are the payoffs. Some of these will use terminology we have not defined yet (notably "continuous functions" and "compact sets"), but all of them are equivalences which only work one way when you make the equivalent statements with sequences.

The ones that extend theorems from the earlier sections are easy to prove given Theorem 6.3. We will not prove the ones that use terminology we have not yet defined in this note, though you are encouraged to revisit them after you know what all the words mean.

Theorem 7.1. *(This extends Theorem 3.6.) Let $(X, \mathcal{T})$ be a topological space, let $A \subseteq X$ and $x \in X$. Then the following are equivalent.*

1. *$x \in \overline{A}$.*
2. *There is a net $w : \mathcal{D} \to A$ such that $w \to x$.*
3. *There is a filter $\mathcal{F}$ on A (by which we mean $\mathcal{F} \subseteq \mathcal{P}(A)$) such that $\mathcal{F} \to x$.*

Theorem 7.2. *(This extends Theorem 3.8.) Let $(X, \mathcal{T})$ be a topological space. Then the following are equivalent.*

1. *$(X, \mathcal{T})$ is Hausdorff.*
2. *Every net in X converges to at most one point.*
3. *Every filter on X converges to at most one point.*

Theorem 7.3. *Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be topological spaces, and $f : X_1 \to X_2$ a function. Then the following are equivalent.*

1. *f is continuous.*
2. *f respects net convergence: if $w : \mathcal{D} \to X_1$ is a net and w converges to x, then $f \circ w : \mathcal{D} \to X_2$ converges to $f(x)$.*
3. *f respects filter convergence: If $\mathcal{F}$ is a filter on X_1 and $\mathcal{F} \to x$, then the filter $f(\mathcal{F}) = \{\, f(F) \subseteq X_2 : F \in \mathcal{F} \,\}$ converges to $f(x)$.*

Theorem 7.4. *Let $(X, \mathcal{T})$ be a topological space. Then the following are equivalent:*

1. *X is compact.*
2. *Every net in X has a convergent subnet.*
3. *Every net in X has an accumulation point.*
4. *Every filter on X has an accumulation point.*
5. *Every filter on X can be extended to a convergent filter.*
6. *Every ultrafilter on X converges.*

Theorem 7.5. *(This extends Theorem 4.8.) Let X be a set and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on X. Then the following are equivalent.*

1. *$\mathcal{T}_1 = \mathcal{T}_2$.*
2. *For every net $w : \mathcal{D} \to X$, w converges in $\mathcal{T}_1$ if and only if it converges in $\mathcal{T}_2$.*
3. *For every filter $\mathcal{F}$ on X, $\mathcal{F}$ converges in $\mathcal{T}_1$ if and only if $\mathcal{F}$ converges in $\mathcal{T}_2$.*

I hope you have enjoyed this trip through nets and filters. Aren't you sad you didn't learn about nets in first year calculus? Oh, speaking of which...

8 Filling in a gap from first year calculus

I promised that we could use nets to fill in a gap from first year calculus. This gap has to do with the definition of integrability. Depending on what level of calculus you took, you might actually have learned this fact, though certainly not with this cleaner terminology.

Recall that a partition of a closed interval $[a, b] \subseteq \mathbb{R}$ is a finite set of points $\{x_0, x_1, x_2, \ldots, x_n\}$ such that

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

Given a partition we can choose a point from each interval of the partition: for each interval $[x_{i-1}, x_i]$ defined by a partition P, choose a point $t_i \in [x_{i-1}, x_i]$. A partition P together with such a choice of points is called a tagged partition. We will denote tagged partitions by an ordered pair:

$$P = (\{x_0, x_1, \ldots, x_n\}, \{t_1, t_2, \ldots, t_n\}) \text{ or simply } P = (\{x_i\}, \{t_j\})$$

Let $\mathcal{P}$ be the collection of all tagged partitions of $[a, b]$.

Now we are going to define an order relation on $\mathcal{P}$. Given two tagged partitions

$$P_1 = (\{x_0, x_1, \ldots, x_n\}, \{t_1, t_2, \ldots, t_n\}) \text{ and } P_2 = (\{y_0, y_1, \ldots, y_m\}, \{s_1, s_2, \ldots, s_m\}),$$

we say that P_2 refines P_1, and denote this by $P_1 \preceq P_2$, if:

1. $\{x_0, x_1, \ldots, x_n\} \subseteq \{y_0, y_1, \ldots, y_m\}$
2. $\{t_1, t_2, \ldots, t_n\} \subseteq \{s_1, s_2, \ldots, s_m\}$

Note that not only is P_2 adding points to the partition part of P_1, it is also not changing any of the points chosen in the tagging in P_1.

With this order, $(\mathcal{P}, \preceq)$ is a directed set. Reflexivity and transitivity are obvious since $\subseteq$ always has those properties. To see directedness, fix two tagged partitions P_1 and P_2 using the same notation as above. We must define a new tagged partition P such that $P_1 \preceq P$ and $P_2 \preceq P$. The set $\{x_0, x_1, \ldots, x_n\} \cup \{y_0, y_1, \ldots, y_m\}$ will (for the most part) serve as the partition for P.

We just have to figure out how to tag this new partition in such a way that it agrees with the tagging of from P_1 and P_2. Every interval in this new partition contains at most two t or s tags (convince yourself of this). If a given interval contains only one, leave it unchanged. If it contains none, add one arbitrarily. If the interval contains two t or s tags, add a new point to the partition to split this interval into two subintervals, one containing each tag. Let P be the tagged partition that results from this procedure. By construction, it is easy to see that P satisfies the definition of directedness.

On to integration. Let $f : [a, b] \to \mathbb{R}$ be a bounded function, and let $\mathcal{P}$ be the collection of all tagged partitions of $[a, b]$ as before. Then define $w : \mathcal{P} \to \mathbb{R}$ by:

$$w : (\{x_0, x_1, \ldots, x_n\}, \{t_1, t_2, \ldots, t_n\}) \mapsto \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}).$$

You may recognize that this assigns to each tagged partition the corresponding Riemann sum. Having read through this note, you should also recognize that w is a net on $\mathbb{R}$.

Definition 8.1. *A function $f : [a, b] \to \mathbb{R}$ is said to be Riemann integrable if the net w defined above converges (in $\mathbb{R}_{usual}$). In this case, the value x that it converges to is denoted by $\int_a^b f(x)\,dx$.*

This is what the somewhat ambiguous "$\lim\limits_{\|P\| \to 0}$" you may have seen was really referring to.

Let's find some familiar things in this rather complicated situation.

First of all, in the case when f is continuous, it has a maximal and minimal value on each closed interval (by the Extreme Value Theorem). Given a partition, we can create tagged partitions in which the chosen t_i's are the points where the minimums of f occur on the corresponding subintervals, and the values of the net on those tagged partitions will be what you know as "lower sums". We can do a similar thing but with maximums to get upper sums.

Perhaps more interestingly, you may recall using dyadic partitions to compute Riemann integrals in first year calculus. At the time, you could only do this if you *already knew* the function was integrable. To be clear:

Fact 8.2. *Let $f : [a,b] \to \mathbb{R}$ be integrable. For each n let P_n be the partition of $[a,b]$ into 2^n subintervals of equal length, that is $P_n = \left\{ a + \frac{k}{2^n}(b-a) \,:\, k = 0,1,\ldots,2^n \right\}$. Then for example*

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} L_f(P_n),$$

where $L_f(P_n)$ denotes the lower sum for P_n. (Any Riemann sum for these partitions will work.)

The reason this works is that the collection of Riemann sums for the tagged partitions corresponding to the lower sums of the P_n's forms a subnet of the net we described earlier. So *if you already know* that the whole net converges, this particular subnet must also converge to the same value. This subnet is just a sequence though, so computing its limit is usually easy and accessible without knowing about nets.