Urysohn's metrization theorem

1 Motivation

By this point in the course, I hope that once you see the statement of Urysohn's metrization theorem you don't feel that it needs much motivating. Having studied metric spaces in detail and having convinced ourselves of how nice they are, a theorem that gives conditions implying that a space is metrizable should seem innately useful.

The techniques used in this proof are also very important ones. One of the biggest payoffs for defining topologies on arbitrary products is that large product spaces (function spaces, specifically) lend themselves very well to embedding things in them. To be more specific, we are going to show a space $(X, \mathcal{T})$ is metrizable by embedding it as a subspace of a metrizable space, specifically $\mathbb{R}^{\mathbb{N}}_{\text{prod}}$.

2 Statement, and preliminary construction

Without further delay, here is the theorem.

Theorem 2.1 (Urysohn metrization theorem)**.** *Every second countable T_3 topological space is metrizable.*

Note that even though second countability, separability, and ccc-ness are equivalent in every metrizable space, you need second countability in this theorem. $\mathbb{R}_{\text{Sorgenfrey}}$ is a space we know that is T_3 and separable but not metrizable. It is also relatively easy to construct a space that is ccc and T_3 but not separable (and therefore not metrizable) by taking a very large product of $(\{0, 1\}, \mathcal{T}_{\text{discrete}})$ with itself. (It should not be obvious that such a space is ccc, but it is.)

We will give two proofs of Urysohn's metrization theorem. One embeds the space into $\mathbb{R}^{\mathbb{N}}_{\text{prod}}$, and the other embeds the space into $\mathbb{R}^{\mathbb{N}}_{\text{unif}}$. Both proofs start with the same initial step, so we do that part first, and then split the proof into two parts.

To begin the proof, let $(X, \mathcal{T})$ be a second countable T_3 space, and let $\mathcal{B} = \{ B_n : n \in \mathbb{N} \}$ be a countable basis for $\mathcal{T}$. We fix this notation until the end of both proofs of the main result.

First, recall Theorem 4.1 from section 9 of the lecture notes (on stronger separation axioms).

Theorem 2.2. *Every regular, second countable topological space is normal.*

So $(X, \mathcal{T})$ is normal. This allows us to use Urysohn's Lemma in the construction we are about to undertake.

Lemma 2.3. *There is a countable collection $\{ f_n : X \to [0, 1] : n \in \mathbb{N} \}$ of continuous functions such that for any $a \in X$ and any open set U containing a, there is some $n \in \mathbb{N}$ such that $f_n(a) > 0$ and $f_n(x) = 0$ for all $x \in X \setminus U$.*

Note that if given a *specific* a and U, it is easy to find a single function that has this property using Urysohn's Lemma (since $\{a\}$ and $X \setminus U$ are disjoint closed sets in this space). The strength of this lemma is that there is a *countable* collection of functions from which you can find a function that does it no matter which point and which set you pick. In a large space there could of course be uncountably many points and sets to separate in this manner, but this lemma says that one of a countable collection of functions can work for all of them.

Proof of Lemma 2.3. Let $A \subseteq \mathbb{N} \times \mathbb{N}$ be the set of ordered pairs of natural numbers (m, n) such that $\overline{B_m} \subseteq B_n$. A is certainly countable, and as we will see shortly it is nonempty.

For each $(m, n) \in A$, apply Urysohn's Lemma to the disjoint closed sets $\overline{B_m}$ and $X \setminus B_n$ to obtain a continuous function $g_{m,n} : X \to [0, 1]$ such that $g_{m,n}(x) = 0$ for all $x \in X \setminus B_n$ and $g_{m,n}(x) = 1$ for all $x \in \overline{B_m}$. Then the collection $\{ g_{m,n} : (m, n) \in A \}$ will be our countable collection of functions. It remains to check that this collection works.

Fix a point $a \in X$ and an open set U containing a. Since $\mathcal{B}$ is a basis, we can find some $n \in \mathbb{N}$ such that $a \in B_n \subseteq U$. Since the space is regular, we can find an open set V such that

$$a \in V \subseteq \overline{V} \subseteq B_n \subseteq U.$$

Finally, again since $\mathcal{B}$ is a basis we can find some $m \in \mathbb{N}$ such that $a \in B_m \subseteq V$. Then we have:

$$a \in B_m \subseteq \overline{B_m} \subseteq \overline{V} \subseteq B_n \subseteq U,$$

and in particular we have $a \in \overline{B_m} \subseteq B_n \subseteq U$. Then the function $g_{m,n}$ defined for this pair of numbers satisfies the requirements of the lemma. □

3 First proof

Proof of Theorem 2.1. As we said, this proof will proceed by constructing a continuous injection $F : X \to \mathbb{R}^{\mathbb{N}}_{\text{prod}}$ that is a homeomorphism onto its range.

Let $\{ f_n : n \in \mathbb{N} \}$ be a collection of functions as in Lemma 2.3. We could of course use the specific set of functions defined in the proof of the Lemma, but from this point onwards we do not need to be that specific. Define $F : X \to \mathbb{R}^{\mathbb{N}}$ by

$$F(x) = (f_1(x), f_2(x), f_3(x), \dots).$$

This function is continuous since each of its component functions is continuous. It is also injective since for any $x \neq y \in X$, we can find an open set U containing x but not y, and therefore we can find an $n \in \mathbb{N}$ such that $f_n(x) > 0$ and $f_n(y) = 0$, and therefore $F(x) \neq F(y)$.

Let $Y = F(X) \subseteq \mathbb{R}^{\mathbb{N}}$ be the range of F. We already know that Y is metrizable (with its subspace topology), since $\mathbb{R}^{\mathbb{N}}_{\text{prod}}$ is metrizable, and we just established that F is continuous and injective. All that remains to show is that F is open.

So fix an open set $U \subseteq X$. We show that $F(U)$ is open in Y. We will do this in the usual way, by fixing an arbitrary point $b \in F(U)$ and finding an open subset V of Y such that $b \in V \subseteq F(U)$.

Let $a = F^{-1}(b) \in U$ (the preimage is a single point since F is injective)w. By the lemma, there is some $N \in \mathbb{N}$ such that $f_N(a) > 0$ and $f_N(x) = 0$ for all $x \in X \setminus U$. We define our open set $V \subseteq Y$ by:

$$V := Y \cap \pi_N^{-1}((0, \infty)).$$

This set is open in the subspace topology on Y since $\pi_N^{-1}((0,\infty))$ is a basic open subset of $\mathbb{R}^{\mathbb{N}}_{\text{prod}}$.

Note that $b \in V$ since

$$\pi_N(b) = \pi_N(F(a)) = f_N(a) > 0.$$

The last thing to show is that $V \subseteq F(U)$. To show this, fix an arbitrary $y \in V$. Since $y \in Y$, we can define $x := F^{-1}(y)$. Since $y \in \pi_N^{-1}((0,\infty))$ we have that

$$0 < \pi_N(y) = \pi_N(F(x)) = f_N(x).$$

But we know that f_N is zero for all points outside of U, and therefore it must be the case that $x \in U$, and in turn that $y = F(x) \in F(U)$, as required.

This shows that $F : X \to Y$ is a homeomorphism, and therefore that X is metrizable. $\square$

Note that we actually embedded X as a subspace of $[0,1]^{\mathbb{N}} \subseteq \mathbb{R}^{\mathbb{N}}$. It should not be so surprising that this is possible, given that we know any metrizable space can be generated by a bounded metric. The space $(0,1)^{\mathbb{N}}$ is homeomorphic to $\mathbb{R}^{\mathbb{N}}$, and so if anything embedding into $[0,1]^{\mathbb{N}}$ gives us *more* room to work with.

4 Second proof

Before we begin, note that we may assume the functions $\{ f_n : n \in \mathbb{N} \}$ provided by Lemma 2.3 actually map into $[0, \frac{1}{n}]$ rather than $[0,1]$, since we can simply replace f_n by $\frac{1}{n} f_n$ if necessary.

Proof of Theorem 2.1. In this proof, we embed X into the "Hilbert cube" $H := \prod_{n\in\mathbb{N}}[0, \frac{1}{n}]$ with the topology generated by the uniform metric. To be clear, let d_u be the metric defined on H by

$$d_u(x,y) = \sup \{ |x_n - y_n| : n \in \mathbb{N} \},$$

where $x = (x_1, x_2, x_3, \dots)$ as usual, and similarly for y. Throughout this proof, we will assume H has the metric topology generated by d_u (or in other words the subspace topology inherited from $\mathbb{R}^{\mathbb{N}}_{\text{unif}}$). Recall also that the uniform topology refines the product topology.

We define $F : X \to H$ in the same way as before:

$$F(x) = (f_1(x), f_2(x), f_3(x), \dots),$$

(where remember that we are assuming f_n maps into $[0, \frac{1}{n}]$ for each n).

Our previous argument also shows that F is injective, and F is still open since the uniform topology refines the product topology. It remains only to show that F is continuous, which is no longer obvious since we are not using the product topology.

We prove that F is continuous using the "continuity at a point" characterization of continuity. So fix $a \in X$, and fix $\epsilon > 0$. We want to find an open set U containing a such that

$$x \in U \quad \Longrightarrow \quad d_u(F(a), F(x)) < \epsilon,$$

or in other words that $F(U) \subseteq B_\epsilon(F(a))$.

This argument proceeds similarly to the proof that $\mathbb{R}^{\mathbb{N}}_{\text{prod}}$ is metrizable, by splitting things up to a finite part we can explicitly take care of, and an infinite tail which is so small that it doesn't matter.

Fix an $N \in \mathbb{N}$ so large that $\frac{1}{N} < \frac{\epsilon}{2}$. Then for each $n \leq N$, since f_n is continuous we can find an open set U_n containing a such that

$$|f_n(a) - f_n(x)| < \frac{\epsilon}{2}$$

for all $x \in U_n$. Then $U := U_1 \cap U_2 \cap \cdots \cap U_N$ is our desired open set. Indeed, if $x \in U$, then by construction $|f_n(x) - f_n(a)| < \frac{\epsilon}{2}$ for all $n \leq N$. On the other hand if $n > N$, then

$$|f_n(x) - f_n(a)| \leq \frac{1}{n} < \frac{1}{N} < \frac{\epsilon}{2}$$

since f_n maps into $[0, \frac{1}{n}]$. In total, we have that $|f_n(x) - f_n(a)| < \frac{\epsilon}{2}$ for all $n \in \mathbb{N}$. Therefore

$$d_u(F(x), F(a)) = \sup \{ \, |f_n(x) - f_n(a)| \, : \, n \in \mathbb{N} \, \} \leq \frac{\epsilon}{2} < \epsilon,$$

as required. $\square$