

14. Arbitrary products

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1 Motivation

In section 8 of the lecture notes we discussed a way of defining a topology on a finite product of topological spaces. That definition more or less agreed with our intuition. We should expect products of open sets in each coordinate space to be open in the product, for example, and the only issue that arises is that these sets only form a basis rather than a topology. So we simply generate a topology from them and call it the product topology. This product topology “played nicely” with the topologies on each of the factors in a number of ways.

In that earlier section, we also saw that every topological property we had studied up to that point other than ccc-ness was finitely productive. Since then we have developed some new topological properties like regularity, normality, and metrizability, and learned that except for normality these are also finitely productive. So, ultimately, most of the properties we have studied are finitely productive. More importantly, the proofs that they are finitely productive have been *easy*. Look back for example to your proof that separability is finitely productive, and you will see that there was almost nothing to do.

However, we saw some hints that when we try to consider infinite products of topological spaces, things become weirder. The two equivalent ways of defining the product topology on a finite product were no longer equivalent. We will soon see that when you take infinite products, the “size” of the product matters. Countable products and uncountable products can act very differently, for example.

2 Background and preliminary definitions

In this section we will recall some definitions from the section on finite products, and formally define an arbitrary product of sets before going on to define topologies on them.

First, recall the following two equivalent definitions of the product topology on a finite product of spaces. The proof of the proposition below is given in section 8 of the lecture notes.

Definition 2.1. *Let (X, \mathcal{T}) and (Y, \mathcal{U}) be topological spaces. The product topology on $X \times Y$ is the topology generated by the basis $\{U \times V : U \in \mathcal{T}, V \in \mathcal{U}\}$.*

More generally if $(X_1, \mathcal{T}_1), \dots, (X_n, \mathcal{T}_n)$ are topological spaces, the product topology on

$$\prod_{i=1}^n X_i = X_1 \times \cdots \times X_n$$

is the topology generated by the basis:

$$\{U_1 \times U_2 \times \cdots \times U_n : U_i \in \mathcal{T}_i \text{ for all } i = 1, \dots, n\}.$$

Proposition 2.2. *Let $(X_1, \mathcal{T}_1), \dots, (X_n, \mathcal{T}_n)$ be topological spaces. Then the product topology on $X_1 \times \dots \times X_n$ is the coarsest topology on $X_1 \times \dots \times X_n$ such that the projection functions π_1, \dots, π_n are continuous.*

An equivalent way of saying this is that the product topology is the one generated by the following subbasis:

$$\mathcal{S} := \{ \pi_1^{-1}(U) : U \in \mathcal{T}_1 \} \cup \dots \cup \{ \pi_n^{-1}(U) : U \in \mathcal{T}_n \}$$

Before we continue, we also need to formally define what an arbitrary product of sets actually is. To do this in the ideal way, we are going to shift how we think about all Cartesian products a little bit.

Notation. Let A, B be sets. We denote the set of all functions $f : A \rightarrow B$ by B^A . For example, the set $\mathbb{R}^{\mathbb{N}}$ is the collection of all sequences of real numbers.

Definition 2.3. *Let I be a nonempty indexing set, and let $\mathcal{X} = \{ X_\alpha : \alpha \in I \}$ be a collection of sets indexed by I . We define the Cartesian product of the sets in \mathcal{X} to be*

$$\prod_{\alpha \in I} X_\alpha := \left\{ f \in \left(\bigcup \mathcal{X} \right)^I : f(\alpha) \in X_\alpha \text{ for all } \alpha \in I \right\}.$$

We use this language because talking about “ I -tuples” is a little ugly, in a way that talking about 7-tuples is not when talking about finite Cartesian products.

This definition extends the usual definition of a finite Cartesian product, so long as for each $n \in \mathbb{N}$, we identify n with the set $\{0, 1, \dots, n-1\}$. This is pretty simple, but ultimately something you have to convince yourself of by staring at the definition of a few minutes. By way of illustration, we discuss \mathbb{R}^2 . When thinking about $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, we use $I = 2 = \{0, 1\}$, and our indexed collection \mathcal{X} simply has two elements $X_0 = X_1 = \mathbb{R}$. Then $\bigcup \mathcal{X} = X_0 \cup X_1 = \mathbb{R}$, and our definition above tells us to think about the set

$$\mathbb{R} \times \mathbb{R} = \mathbb{R}^{\{0,1\}}.$$

That is, we think of \mathbb{R}^2 as the collection of all functions $f : \{0, 1\} \rightarrow \mathbb{R}$. This can be identified with the usual set of ordered pairs of real numbers via the correspondences $f \mapsto (f(0), f(1))$, and $(a, b) \mapsto f_{(a,b)}$, where $f_{(a,b)} : \{0, 1\} \rightarrow \mathbb{R}$ is defined by $f_{(a,b)}(0) = a$ and $f_{(a,b)}(1) = b$. Essentially, we identify a function $f : \{0, 1\} \rightarrow \mathbb{R}$ with the ordered list of its outputs.

This way of thinking about Cartesian products feels a bit clunky in the finite case, but it is all we have to work with in the case of arbitrary indexed families.

Exercise 2.4. Let I be a nonempty indexing set, and let X be a nonempty set. Let $\mathcal{X} = \{ X_\alpha : \alpha \in I \}$ be the indexed family of sets such that $X_\alpha = X$ for all $\alpha \in I$. Show that $\prod_{\alpha \in I} X_\alpha = X^I$.

Exercise 2.5. Fully convince yourself that this definition of Cartesian products extends the usual definition of “ordered n -tuples” for finite products.

Exercise 2.6. Let I be a nonempty indexing set, and let $\mathcal{X} = \{X_\alpha : \alpha \in I\}$ be a collection of sets such that $X_\alpha = \emptyset$ for at least one $\alpha \in I$. Show that $\prod_{\alpha \in I} X_\alpha = \emptyset$.

Exercise 2.7. Using the identification $n = \{0, 1, \dots, n-1\}$ we mentioned above, show that $n^k = |n^k|$. What we mean here is that on the left side is usual exponentiation of natural numbers, while on the right side we have the cardinality of the set of all functions $f : k \rightarrow n$.

This is the most general way to define exponentiation of natural numbers, in the sense that it generalizes to infinite sets and cardinal numbers.

Finally, before we proceed to defining topologies we will formally redefine projection functions in this new setting.

Definition 2.8. Let I be a nonempty indexing set, and let $\mathcal{X} = \{X_\alpha : \alpha \in I\}$ be a collection of nonempty sets. Let $X = \prod_{\alpha \in I} X_\alpha$ be their Cartesian product. For each $\alpha \in I$, define the projection map $\pi_\alpha : X \rightarrow X_\alpha$ by $\pi_\alpha(f) = f(\alpha)$.

We again remark that this definition extends the usual way we think about \mathbb{R}^2 . Let π_0 and π_1 denote the usual projection maps onto the x - and y -axes, respectively, and fix an ordered pair $(a, b) \in \mathbb{R}^2$. In our earlier context, we knew that $\pi_0(a, b) = a$. This is also true in our new context. For $f \in \mathbb{R}^2$, by definition, $\pi_0(f) = f(0)$. By the identification described above, the ordered pair identified with f is $(f(0), f(1))$, and so this works as we expect.

Exercise 2.9. Fully convince yourself that this definition of projection functions extends the definition of projection functions we already had for finite products.

3 The space $\mathbb{R}_{\text{box}}^{\mathbb{N}}$

Before we define things in complete generality, we will treat the much more familiar special case of the countably infinite product of copies of \mathbb{R} . That is, we will discuss $\prod_{n \in \mathbb{N}} \mathbb{R}$, which by Exercise 2.4 above simply equals $\mathbb{R}^{\mathbb{N}}$, the collection of all sequences of real numbers. In this section, we will always assume \mathbb{R} has its usual topology, and we will always use $x = (x_1, x_2, x_3, \dots)$ and $y = (y_1, y_2, y_3, \dots)$ as generic points of $\mathbb{R}^{\mathbb{N}}$.

When visualizing this set, you should think of drawing an infinite sequence of parallel vertical lines (each representing a copy of \mathbb{R}), and you should think of an element of $\mathbb{R}^{\mathbb{N}}$ as a choice of one point from each line, or as the “path” formed by joining each chosen point up with a line.

To illustrate the subtlety of infinite products, we will directly transfer the two definitions of the product topology on finite products stated in the previous section, and show that they are different. In this section in particular, we define a topology with an idea exactly analogous to Definition 2.1.

Definition 3.1. Let \mathcal{T}_{box} be the topology on $\mathbb{R}^{\mathbb{N}}$ generated by the following basis

$$\mathcal{B} = \left\{ \prod_{n \in \mathbb{N}} U_n : U_n \subseteq \mathbb{R} \text{ is open} \right\}.$$

\mathcal{T}_{box} is called the box topology. We denote $(\mathbb{R}^{\mathbb{N}}, \mathcal{T}_{\text{box}})$ by $\mathbb{R}_{\text{box}}^{\mathbb{N}}$. To be clear, a basic open set in \mathcal{T}_{box} is a set of the form

$$U_1 \times U_2 \times U_3 \times U_4 \times \cdots,$$

where U_n is an open subset of \mathbb{R} for all n .

So for example, the following sets would be basic open in this topology:

$$\begin{aligned} &(0, 1) \times (0, 1) \times (0, 1) \times \cdots \\ &(-1, 1) \times \left(-\frac{1}{2}, \frac{1}{2}\right) \times \left(-\frac{1}{3}, \frac{1}{3}\right) \times \cdots \\ &\mathbb{R} \times \mathbb{R} \times (0, 1) \times (7, 8) \times (7, 8) \times (7, 8) \times \cdots \end{aligned}$$

This is the natural-feeling way to extend the previous definition of the product topology. However, this topology has a lot of undesirable properties. The problems stem from the fact that we essentially have *too much* freedom to define open sets in this space. We analyze some properties of this space below. Be sure to draw pictures as we go.

Proposition 3.2. $\mathbb{R}_{\text{box}}^{\mathbb{N}}$ is Hausdorff.

Proof. Let $x, y \in \mathbb{R}^{\mathbb{N}}$. If $x \neq y$, then it must be that $x_n \neq y_n$ for some $n \in \mathbb{N}$. Since \mathbb{R} is Hausdorff, find disjoint open sets V_1 and V_2 such that $x_n \in V_1$ and $y_n \in V_2$. Then the open sets

$$U_1 = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \times V_1 \times \mathbb{R} \times \cdots \quad \text{and} \quad U_2 = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \times V_2 \times \mathbb{R} \times \cdots$$

(where the non- \mathbb{R} sets occur in the n^{th} coordinates of both sets) are disjoint open subsets of $\mathbb{R}_{\text{box}}^{\mathbb{N}}$ containing x and y respectively.

(Note, for future reference, that all but one of the coordinates of U_1 and U_2 are all of \mathbb{R} . This will come up later.) □

This is simple enough. In fact, it is easy to show that $\mathbb{R}_{\text{box}}^{\mathbb{N}}$ is regular, and in fact completely regular. For reference

Proposition 3.3. $\mathbb{R}_{\text{box}}^{\mathbb{N}}$ is regular.

Proof. Exercise. This will be on the Big List. □

Now on to some countability properties.

Proposition 3.4. $\mathbb{R}_{\text{box}}^{\mathbb{N}}$ is not first countable.

Proof. This is a classic example of what is often called a “diagonalization” proof, similar to Cantor’s famous diagonalization argument.

We will show that the constant zero function $\bar{0} = (0, 0, 0, \dots)$ does not have a countable local basis. Let $\mathcal{B}_{\bar{0}} = \{V_n : n \in \mathbb{N}\}$ be a countable collection of open subsets of $\mathbb{R}_{\text{box}}^{\mathbb{N}}$ that each contain $\bar{0}$. We will show that $\mathcal{B}_{\bar{0}}$ is not a local basis.

Shrinking them if necessary, we may assume without loss of generality that each V_n is a basic open set. So for each V_n , there is a sequence $\epsilon_{n,1}, \epsilon_{n,2}, \epsilon_{n,3}, \dots > 0$ such that

$$V_n = B_{\epsilon_{n,1}}(0) \times B_{\epsilon_{n,2}}(0) \times B_{\epsilon_{n,3}}(0) \times \dots$$

(we are using ball notation here because the alternative is writing $(-\epsilon_{n,1}, \epsilon_{n,1})$ every time, which looks more confusing).

Let $U \subseteq \mathbb{R}^{\mathbb{N}}$ be defined by:

$$U = B_{\frac{\epsilon_{1,1}}{2}}(0) \times B_{\frac{\epsilon_{2,2}}{2}}(0) \times B_{\frac{\epsilon_{3,3}}{2}}(0) \times B_{\frac{\epsilon_{4,4}}{2}}(0) \times \dots$$

Obviously $\bar{0} \in U$, but $V_n \not\subseteq U$ for any n , and therefore $\mathcal{B}_{\bar{0}}$ is not a local basis at 0. This is because for any n the n^{th} coordinate of V_n is the set $B_{\epsilon_{n,n}}(0)$, which is not contained in the n^{th} coordinate of U which is the set $B_{\frac{\epsilon_{n,n}}{2}}(0)$. \square

This result also implies that $\mathbb{R}_{\text{box}}^{\mathbb{N}}$ is both not second countable and not metrizable, which already means it is not as nice as we would hope. It turns out to not be separable or ccc either.

Proposition 3.5. $\mathbb{R}_{\text{box}}^{\mathbb{N}}$ is not ccc, and therefore not separable.

Proof. Exercise.

(Hint: Let $x \in \mathbb{Z}^{\mathbb{N}} \subseteq \mathbb{R}^{\mathbb{N}}$ be a sequence with all integer values. Find an open set around x that contains no other sequences with all integer values. Recall (or prove) that $\mathbb{Z}^{\mathbb{N}}$ is uncountable to finish the proof.) \square

When talking about how separative this space is, we never mentioned normality. Astonishingly, it is not known whether $\mathbb{R}_{\text{box}}^{\mathbb{N}}$ is normal. We know that if you assume the Continuum Hypothesis you can prove it is normal, but no one knows if it’s normal in ZFC.

In conclusion, this space is weird. We should hope that $\mathbb{R}^{\mathbb{N}}$ has a nice topology, since \mathbb{R} has such a nice topology. We should certainly expect to *be able to prove* that the topology on $\mathbb{R}^{\mathbb{N}}$ is normal or not normal. The solution is to use the generalization of the other definition of the finite product topology.

4 The space $\mathbb{R}_{\text{prod}}^{\mathbb{N}}$

In this section again, we will always assume \mathbb{R} has its usual topology, and we will always use $x = (x_1, x_2, x_3, \dots)$ and $y = (y_1, y_2, y_3, \dots)$ as generic points of $\mathbb{R}^{\mathbb{N}}$. We will also always use $\pi_n : \mathbb{R}^{\mathbb{N}} \rightarrow \mathbb{R}$ to denote the usual projection functions $\pi_n(x) = x_n$.

This topology is formed by generalizing the idea in Proposition 2.2. First, we analyze the preimages of open sets under projections on their own.

Let $U \subseteq \mathbb{R}$ be a subset. Then its preimage $\pi_n^{-1}(U)$ under the n^{th} projection function is the set

$$\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \times U \times \mathbb{R} \times \cdots,$$

where the U is in the n^{th} coordinate. In other words, $\pi_n^{-1}(U)$ is the collection of all sequences of real numbers whose n^{th} term is in U . Even more specifically, for example, $\pi_1^{-1}((0, 1))$ is the collection of all sequences of real numbers whose first term is between 0 and 1.

Note that by intersecting more than one of these sets, you can add more than one restriction to these sequences. For example, the set

$$\pi_1^{-1}((0, 1)) \cap \pi_3^{-1}((7, 8)) \cap \pi_7^{-1}(\mathbb{Q})$$

is the collection of sequences of real numbers whose first term is between 0 and 1, whose third term is between 7 and 8, and whose seventh term is rational.

Take a minute to play around with these preimages and get a feeling for what these sorts of sets look like.

Definition 4.1. The product topology on $\mathbb{R}^{\mathbb{N}}$, denoted $\mathcal{T}_{\text{prod}}$, is the coarsest topology on $\mathbb{R}^{\mathbb{N}}$ such that $\pi_n : \mathbb{R}^{\mathbb{N}} \rightarrow \mathbb{R}$ is continuous for all $n \in \mathbb{N}$.

Equivalently, for each $n \in \mathbb{N}$, let

$$\mathcal{S}_n := \{ \pi_n^{-1}(U) : U \subseteq \mathbb{R} \text{ open} \}.$$

Then $\mathcal{S} := \bigcup_{n \in \mathbb{N}} \mathcal{S}_n$ is a subbasis on $\mathbb{R}^{\mathbb{N}}$, and the topology it generates is $\mathcal{T}_{\text{prod}}$. We denote $(\mathbb{R}^{\mathbb{N}}, \mathcal{T}_{\text{prod}})$ by $\mathbb{R}_{\text{prod}}^{\mathbb{N}}$ in this document, and most of the time in later sets of notes we will simply write $\mathbb{R}^{\mathbb{N}}$.

Exercise 4.2. Justify the use of the word “equivalently” in the previous definition.

This is the formal definition, but with a little bit of analysis we can find an easier way of thinking about this. The \mathcal{S} defined above is a subbasis, meaning the collection of all finite intersections of elements of \mathcal{S} forms a basis. What do these finite intersections look like?

Well, we basically answered this question when we were thinking about preimages of projection functions above. When you intersect finitely many sets of the form $\pi_{n_1}^{-1}(U_1), \dots, \pi_{n_k}^{-1}(U_k)$, you get the set whose n_i^{th} coordinate is U_i for all $i = 1, \dots, k$, and all of whose other coordinates are all of \mathbb{R} .

This allows us to state a definition of the product topology that is easier to work with, and much easier to picture.

Definition 4.3. $\mathcal{T}_{\text{prod}}$ is the topology on $\mathbb{R}^{\mathbb{N}}$ generated by the basis

$$\mathcal{B}_{\text{prod}} = \left\{ \prod_{n \in \mathbb{N}} U_n : U_n \subseteq \mathbb{R} \text{ is open, and } U_n = \mathbb{R} \text{ for all but finitely many } n \right\}$$

No matter which formulation appeals most appeals to you, this topology turns out to be the “correct” way of defining a topology on $\mathbb{R}^{\mathbb{N}}$ that extends the properties of the usual topology as naturally as we would like. We start with some easy facts.

Proposition 4.4. $\mathcal{T}_{\text{prod}} \subseteq \mathcal{T}_{\text{box}}$.

Proof. This is immediate from the definition. In particular though, note that this containment is strict. For example the set $(0, 1) \times (0, 1) \times (0, 1) \times \cdots$ is open in \mathcal{T}_{box} but not in $\mathcal{T}_{\text{prod}}$. \square

Proposition 4.5. $\mathbb{R}_{\text{prod}}^{\mathbb{N}}$ is Hausdorff.

Proof. The same proof as in Proposition 3.2 works here. \square

It also turns out that $\mathbb{R}_{\text{prod}}^{\mathbb{N}}$ is regular and normal, but we will see this later.

Proposition 4.6. $\mathbb{R}_{\text{prod}}^{\mathbb{N}}$ is first countable.

Proof. Exercise. (You will do this on the Big List. The proof is related to the fact that $\text{Fin}(\mathbb{N}) = \{A \subseteq \mathbb{N} : A \text{ is finite}\}$ is countable, which you proved in section 4 of the Big List.) \square

Proposition 4.7. $\mathbb{R}_{\text{prod}}^{\mathbb{N}}$ is separable.

Proof. We have to be a little bit careful here. We are tempted to say that $\mathbb{Q}^{\mathbb{N}} \subseteq \mathbb{R}^{\mathbb{N}}$ should be dense, but $\mathbb{Q}^{\mathbb{N}}$, the set of all sequences of rational numbers, is uncountable. We can see this easily since every real number is the limit of a sequence of rationals.

To fix this, for each $n \in \mathbb{N}$, define

$$D_n = \left(\prod_{i=1}^n \mathbb{Q} \right) \times \left(\prod_{i=n+1}^{\infty} \{0\} \right) = \left\{ x \in \mathbb{Q}^{\mathbb{N}} : x_k = 0 \text{ for all } k > n \right\}.$$

That is, D_n is the collection of all sequences of rationals that are constantly 0 after their n^{th} term. Let $D = \bigcup_{n \in \mathbb{N}} D_n$.

We first note that D is countable. Each D_n is countable, since it can easily be put into bijection with \mathbb{Q}^n . For convenience, we show this for D_2 . D_2 is the set of all sequence of the form $q_1, q_2, 0, 0, 0, \dots$ where $q_1, q_2 \in \mathbb{Q}$. Define $f : D_2 \rightarrow \mathbb{Q}^2$ by

$$f : (q_1, q_2, 0, 0, \dots) \mapsto (q_1, q_2).$$

This map is clearly a bijection and \mathbb{Q}^2 is countable, thus D_2 is also countable. In an analogous way, we can see that D_n is countable for all n , and therefore D is a countable union of countable sets.

We now show that D is dense in $\mathbb{R}_{\text{prod}}^{\mathbb{N}}$. Let $U = \prod_{n \in \mathbb{N}} U_n$ be a basic open subset of $\mathbb{R}^{\mathbb{N}}$. Then as we saw, $U_n = \mathbb{R}$ for all but finitely many n . So let N be such that $U_k = \mathbb{R}$ for all $k > N$. Then $U \cap D_N \neq \emptyset$. To see this, on the one hand we have that $\mathbb{Q} \cap U_k \neq \emptyset$ for all $k = 1, \dots, N$ since \mathbb{Q} is dense in \mathbb{R} . On the other hand, $0 \in U_k = \mathbb{R}$ for all $k > N$. \square

Okay, this is looking much better. In fact, this space is almost as nice as we could ask for.

Proposition 4.8. $\mathbb{R}_{\text{prod}}^{\mathbb{N}}$ is metrizable.

Proof. There are several metrics we can use to show this (just as there were several metrics we could have used to show that finite products of metrizable spaces are metrizable), but most of them are pretty similar in flavour. We present two of them here.

First of all, let \bar{d} be a metric that generates the usual topology on \mathbb{R} and is bounded by 1. Two examples of such metrics were given in the Big List. For our purposes here, you can imagine that $\bar{d}(x, y) = \min\{1, |x - y|\}$, but all that matters is that it is bounded by 1.

Define two metrics d_1 and d_2 on $\mathbb{R}^{\mathbb{N}}$ by

$$d_1(x, y) = \sup \left\{ \frac{\bar{d}(x_n, y_n)}{n} : n \in \mathbb{N} \right\} \quad \text{and} \quad d_2(x, y) = \sum_{n=1}^{\infty} \frac{\bar{d}(x_n, y_n)}{2^n}$$

Note first that both of these are well-defined. The first one is well-defined since the range of \bar{d} is bounded, and the second one is well-defined by the Basic Comparison Test for series, because

$$0 \leq \frac{\bar{d}(x_n, y_n)}{2^n} \leq \frac{1}{2^n},$$

and $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges.

Exercise 4.9. Show that d_1 and d_2 are metrics on $\mathbb{R}^{\mathbb{N}}$.

We will do the proof that $\mathbb{R}_{\text{prod}}^{\mathbb{N}}$ is metrizable using d_1 . So for the remainder of this proof, we denote ϵ -balls in $\mathbb{R}^{\mathbb{N}}$ according to d_1 by $B_{\epsilon}(x)$, and we will denote ϵ -balls in \mathbb{R} according to \bar{d} by $C_{\epsilon}(x)$.

Let \mathcal{B}_1 be the basis of open balls generated by the metric d_1 , and let $\mathcal{B}_{\text{prod}}$ be the basis described in Definition 4.3. We want to apply Corollary 4.2 from the lecture notes on bases to check that \mathcal{B}_1 generates the product topology.

We must first show that every open ball in \mathcal{B}_1 is open in the product topology. Let $B_{\epsilon}(x)$ be such a ball, and let $y \in B_{\epsilon}(x)$ be an arbitrary point. Then there is some $\delta > 0$ such that $y \in B_{\delta}(y) \subseteq B_{\epsilon}(x)$. (Technically this δ could be anything less than $\epsilon - d_1(y, x)$, but its precise value is not important.) If we find a basic open set U from the product topology such that $y \in U \subseteq B_{\delta}(y) \subseteq B_{\epsilon}(x)$, this will establish that $B_{\epsilon}(x)$ is open in the product topology.

Let $N \in \mathbb{N}$ be so large that $\frac{1}{N} < \frac{\delta}{2}$. For $n < N$, define $U_n = C_{\frac{1}{N}}(y_n)$, and for $n \geq N$, define $U_n = \mathbb{R}$. We claim that the set $U = \prod_{n \in \mathbb{N}} U_n$ works. Obviously $U \in \mathcal{B}_{\text{prod}}$, since $U_n = \mathbb{R}$ for all but finitely many n . Now fix $z = (z_1, z_2, \dots) \in U$. If $n < N$, then $z_n \in C_{\frac{1}{N}}(y_n)$, and so

$$\bar{d}(y_n, z_n) < \frac{1}{N} \implies \frac{1}{n} \bar{d}(y_n, z_n) < \frac{1}{nN} \leq \frac{1}{N} < \frac{\delta}{2}.$$

On the other hand, if $n \geq N$, we have:

$$\frac{1}{n} \bar{d}(y_n, z_n) \leq \frac{1}{n} \leq \frac{1}{N} < \frac{\delta}{2},$$

where the first inequality is because \bar{d} is bounded by 1. In summary, we have that

$$d_1(y, z) := \sup \left\{ \frac{\bar{d}(y_n, z_n)}{n} : n \in \mathbb{N} \right\} \leq \frac{\delta}{2} < \delta.$$

This shows that $z \in B_\delta(y)$, completing the proof that every \bar{d} -ball is open.

To show the second property required by the corollary we are using, let $U = \prod_{n \in \mathbb{N}} U_n \in \mathcal{B}_{\text{prod}}$ be a basic open set of the product topology, and let $x \in U$. We want to find an $\epsilon > 0$ such that $B_\epsilon(x) \subseteq U$. By definition of $\mathcal{B}_{\text{prod}}$, there is a finite set $F \subseteq \mathbb{N}$ such that $U_n = \mathbb{R}$ for all $n \in \mathbb{N} \setminus F$. For every $n \in F$, by definition of the metric topology on \mathbb{R} there is an $\epsilon_n > 0$ such that $x_n \in C_{\epsilon_n}(x_n) \subseteq U_n$. Let $\epsilon = \frac{1}{2} \min \left\{ \frac{\epsilon_n}{n} : n \in F \right\}$, which exists and is positive since F is finite.

We now check that this ϵ works, which is to say that $B_\epsilon(x) \subseteq U$. To see this, fix any $y \in B_\epsilon(x)$. If $n \in F$, we have that

$$\frac{1}{n} \bar{d}(x_n, y_n) \leq \sup \left\{ \frac{\bar{d}(x_k, y_k)}{k} : k \in \mathbb{N} \right\} = d_1(x, y) < \epsilon < \frac{1}{n} \epsilon_n,$$

and in particular $\bar{d}(x_n, y_n) < \epsilon_n$. Therefore $y_n \in C_{\epsilon_n}(x_n) \subseteq U_n$. If $n \notin F$, then $y_n \in U_n = \mathbb{R}$ obviously. Therefore $y_n \in U_n$ for all $n \in \mathbb{N}$, or in other words $y \in U$. \square

Exercise 4.10. Do the previous proof but using d_2 instead of d_1 . Some of the details are slightly different, but the ideas are all the same.

As we learned in the section on metric spaces, three of the countability properties are equivalent in a metric space. This gives us some results for free.

Corollary 4.11. $\mathbb{R}_{\text{prod}}^{\mathbb{N}}$ is second countable and ccc.

Proof. We know that $\mathbb{R}_{\text{prod}}^{\mathbb{N}}$ is metrizable and separable, and that a metrizable space is separable iff it is second countable iff it is ccc. \square

Also for free:

Corollary 4.12. $\mathbb{R}_{\text{prod}}^{\mathbb{N}}$ is regular and normal.

By this point, you should be convinced that the product topology is a *much* more desirable topology on $\mathbb{R}^{\mathbb{N}}$ than the box topology, despite the box topology being easier to describe. We have explored the results about $\mathbb{R}_{\text{prod}}^{\mathbb{N}}$ in great detail in this section, in hopes that the more general results about arbitrary products to come will be less intimidating.

Before we move on, we take a moment to discuss continuous functions and convergent sequences. Just as in the case of finite products, these things act *exactly* the way we hope they would.

Proposition 4.13. *Let $\{y_n\}_{n \in \mathbb{N}}$ be a sequence in $\mathbb{R}_{\text{prod}}^{\mathbb{N}}$. Then the sequence converges to a point $x = (x_1, x_2, x_3, \dots)$ if and only if the “coordinate sequence” $\{\pi_k(y_n)\}_{n \in \mathbb{N}}$ in \mathbb{R} converges to x_k for all $k \in \mathbb{N}$.*

Proof. Exercise. This is much easier than it looks, as long as you understand what the open sets in the product topology look like. \square

Note that this is *not* true in the box topology. For example the sequence $\{x_n\}_{n \in \mathbb{N}}$ defined by:

$$x_n = (0, 0, \dots, 0, 0, 1, 0, 0, \dots)$$

(where the 1 is in the n^{th} coordinate) converges to $\bar{0}$ in the product topology since each coordinate sequence is constantly 0 except for one term, but does not converge in the box topology since for example the set

$$U = (-1, 1) \times (-1, 1) \times (-1, 1) \times \dots$$

contains $\bar{0}$ but contains no element of the sequence.

Proposition 4.14. *Let (X, \mathcal{T}) be a topological space, and let $f : X \rightarrow \mathbb{R}^{\mathbb{N}}$ be a function, where $\mathbb{R}^{\mathbb{N}}$ has the product topology. Let the coordinate functions of f be called f_n , for $n \in \mathbb{N}$, so that for $x \in X$*

$$f(x) = (f_1(x), f_2(x), f_3(x), \dots)$$

Then f is continuous if and only if $f_n : X \rightarrow \mathbb{R}$ is continuous for every $n \in \mathbb{N}$.

Proof. Note that $f_n = \pi_n \circ f$, so by definition of the product topology the (\Rightarrow) direction is immediate. The (\Leftarrow) direction is left as an exercise. As a hint for this exercise, do it by proving that the preimage of *subbasic* open sets is open, using the subbasis of $\mathbb{R}_{\text{prod}}^{\mathbb{N}}$ given in [the definition of the product topology](#). \square

Again, note that this is not true of the box topology. For example the function $f : \mathbb{R} \rightarrow \mathbb{R}^{\mathbb{N}}$ given by

$$f(t) = (t, t, t, t, \dots)$$

is continuous in the product topology (by the proposition), but not continuous in the box topology. This is a classic example. To see that it is not continuous in the box topology, compute the preimage of the basic open set:

$$U = (-1, 1) \times \left(-\frac{1}{2}, \frac{1}{2}\right) \times \left(-\frac{1}{3}, \frac{1}{3}\right) \times \dots$$

5 The uniform topology on $\mathbb{R}^{\mathbb{N}}$, and completeness

While we were defining the metric d_1 in the previous section, we noted that it is well-defined because the range of \bar{d} is bounded. You may have thought to yourself that the following would also be well-defined:

$$d_u(x, y) = \sup \{ \bar{d}(x_n, y_n) : n \in \mathbb{N} \},$$

and you would be right to think so. This is well-defined and is a metric on $\mathbb{R}^{\mathbb{N}}$, whose definition should remind us of the square metric on \mathbb{R}^2 . This is called the uniform metric, and the topology $\mathcal{T}_{\text{unif}}$ on $\mathbb{R}^{\mathbb{N}}$ it generates is called the uniform topology. We will write $\mathbb{R}_{\text{unif}}^{\mathbb{N}}$ to represent $\mathbb{R}^{\mathbb{N}}$ with this topology. Given an element $x \in \mathbb{R}^{\mathbb{N}}$, an ϵ -ball around x looks a bit like, but not exactly like, a “tube” of radius ϵ centred on the sequence x .

This space is obviously metrizable (and therefore Hausdorff, regular, normal, and first countable).

Exercise 5.1. Show that $\mathcal{T}_{\text{prod}} \subseteq \mathcal{T}_{\text{unif}} \subseteq \mathcal{T}_{\text{box}}$.

Exercise 5.2. Show that both of the containments in the previous exercise are strict (by explicitly specifying open sets that witness this).

Proposition 5.3. $\mathbb{R}_{\text{unif}}^{\mathbb{N}}$ is not ccc.

Proof. **Exercise.**

(Hint: The same idea as in the [proof that \$\mathbb{R}_{\text{box}}^{\mathbb{N}}\$ is not ccc](#) will work here.) □

Corollary 5.4. $\mathbb{R}_{\text{unif}}^{\mathbb{N}}$ is not separable or second countable.

Proof. Separability, second countability and ccc-ness are all equivalent in a metrizable space. □

So this is a fine metric, but why is it relevant? The answer has to do with completeness of metric spaces. We will discuss this in some more detail during the general case later, but for now we will analyze the product and uniform topologies on $\mathbb{R}^{\mathbb{N}}$ from the point of view of completeness (the box topology is not even metrizable, so it does not make sense to talk about completeness in that space).

Proposition 5.5. $(\mathbb{R}^{\mathbb{N}}, d_1)$ and $(\mathbb{R}^{\mathbb{N}}, d_u)$ are both complete metric spaces.

Proof. We treat the product topology first. Let $\{y_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in $\mathbb{R}^{\mathbb{N}}$ according to the metric d_1 (which generates the product topology). We want to show that it converges. For any fixed $i \in \mathbb{N}$, we have the coordinate sequence $\{\pi_i(y_n)\}_{n \in \mathbb{N}}$ is a Cauchy sequence in \mathbb{R} according to the metric \bar{d} . This is because:

$$\begin{aligned} d_1(y_n, y_m) &= \sup \left\{ \frac{\bar{d}(\pi_k(y_n), \pi_k(y_m))}{k} : k \in \mathbb{N} \right\} \geq \left(\frac{1}{i} \right) \bar{d}(\pi_i(y_n), \pi_i(y_m)) \\ &\Rightarrow \bar{d}(\pi_i(y_n), \pi_i(y_m)) \leq i d_1(y_n, y_m). \end{aligned}$$

Since \mathbb{R} is complete with its usual metric (and therefore also with \bar{d}), this means each coordinate sequence converges, and therefore $\{y_n\}_{n \in \mathbb{N}}$ converges by Proposition 4.13.

Now, assume $\{y_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}^{\mathbb{N}}$ according to the metric d_u . As in the previous case, we have that

$$\bar{d}(\pi_i(y_n), \pi_i(y_m)) \leq d_u(y_n, y_m),$$

although in this case the inequality is just immediate from the definition of d_u . This means each coordinate sequence is a Cauchy sequence in \mathbb{R} (with respect to \bar{d}) and therefore converges. Let $x_i \in \mathbb{R}$ be the point to which $\{\pi_i(y_n)\}_{n \in \mathbb{N}}$ converges. It remains to show that $\{y_n\}_{n \in \mathbb{N}}$ converges to $x = (x_1, x_2, x_3, \dots)$ in $\mathbb{R}_{\text{unif}}^{\mathbb{N}}$.

So, fix $\epsilon > 0$. Using the fact that our sequence is Cauchy, let $N \in \mathbb{N}$ be so large that $d_u(y_n, y_m) < \frac{\epsilon}{2}$ for all $n, m > N$. Then in particular we have that $\bar{d}(\pi_i(y_n), \pi_i(y_m)) < \frac{\epsilon}{2}$ for all $i \in \mathbb{N}$. Holding n and i fixed and taking a limit in m , we have:

$$\bar{d}(\pi_i(y_n), x_i) = \lim_{m \rightarrow \infty} \bar{d}(\pi_i(y_n), \pi_i(y_m)) \leq \frac{\epsilon}{2}.$$

(We can do this since the projection functions and the metric are continuous.) This inequality is true for all i , and all $n \geq N$. Therefore for all such n , we have

$$d_u(y_n, x) = \sup \{ \bar{d}(\pi_i(y_n), x_i) : i \in \mathbb{N} \} \leq \frac{\epsilon}{2} < \epsilon$$

and so the tail of the sequence after N is in the ϵ -ball around x , as required. \square

As we said, completeness is the reason we introduce this uniform metric. We will shortly see that an *arbitrary* product of complete metric spaces with its product topology need not even be metrizable, let alone complete. However, an arbitrary product of complete metric spaces with its uniform metric (which can be defined the same way in general) is always complete.

When we discussed completeness the first time, we saw that it can depend on the metric being used (and therefore was not a topological invariant). However, there is a topological invariant lurking around completeness that we can define:

Definition 5.6. *A topological space (X, \mathcal{T}) is completely metrizable (or sometimes topologically complete) there exists a metric d that generates \mathcal{T} and such that (X, d) is complete.*

Obviously every completely metrizable space is metrizable, but the converse is not true. The classic example is \mathbb{Q} with its usual subspace topology. This is metrizable, as we know, but not completely metrizable due to the Baire Category Theorem. (This theorem can be found at the end of the Big List section on metric spaces, and the proof that \mathbb{Q} is not completely metrizable is easy from that theorem. Try it!)

6 Summary of results about $\mathbb{R}^{\mathbb{N}}$

It is easy to get lost in all of these results about different topologies on $\mathbb{R}^{\mathbb{N}}$, so we summarize them here. Any results below that are proved in the notes above link to the corresponding results.

	$\mathbb{R}_{\text{prod}}^{\mathbb{N}}$	$\mathbb{R}_{\text{unif}}^{\mathbb{N}}$	$\mathbb{R}_{\text{box}}^{\mathbb{N}}$
Hausdorff	✓	✓	✓
regular	✓	✓	✓
normal	✓	✓	???
first countable	✓	✓	✗
metrizable	✓	✓	✗
completely metrizable	✓	✓	✗
second countable	✓	✗	✗
separable	✓	✗	✗
ccc	✓	✗	✗

7 Arbitrary products

Having studied $\mathbb{R}^{\mathbb{N}}$ in *great detail*, our treatment of arbitrary products will be much quicker.

Definition 7.1. Let I be a nonempty indexing set, and let $\mathcal{X} = \{(X_\alpha, \mathcal{T}_\alpha) : \alpha \in I\}$ be a collection of nonempty topological spaces indexed by I . Let $X = \prod_{\alpha \in I} X_\alpha$ be the Cartesian product of the underlying sets in \mathcal{X} . The topology \mathcal{T}_{box} on X generated by the basis:

$$\mathcal{B}_{\text{box}} = \left\{ \prod_{\alpha \in I} U_\alpha : U_\alpha \subseteq X_\alpha \text{ is open} \right\}$$

is called the box topology.

Definition 7.2. Let I , \mathcal{X} , and X be as in the previous definition. The topology $\mathcal{T}_{\text{prod}}$ on X is the coarsest topology on X such that the projection $\pi_\alpha : X \rightarrow X_\alpha$ is continuous for all $\alpha \in I$.

Equivalently, for each $\alpha \in I$, let

$$\mathcal{S}_\alpha = \{ \pi_\alpha^{-1}(U) : U \subseteq X_\alpha \text{ is open} \}.$$

Then $\mathcal{S} := \bigcup_{\alpha \in I} \mathcal{S}_\alpha$ is a subbasis on X , and the topology it generates is $\mathcal{T}_{\text{prod}}$.

Just as we had for $\mathbb{R}_{\text{prod}}^{\mathbb{N}}$, the following characterization of the product topology on X is easier to work with.

Definition 7.3. Again, let I , \mathcal{X} , and X be as in the previous definitions. $\mathcal{T}_{\text{prod}}$ is the topology on X generated by the basis

$$\mathcal{B}_{\text{prod}} = \left\{ \prod_{\alpha \in I} U_\alpha : U_\alpha \subseteq X_\alpha \text{ is open, and } U_\alpha = X_\alpha \text{ for all but finitely many } \alpha \in I \right\}$$

Whenever we refer to a product of topological spaces, the reader should assume we are using the product topology unless otherwise specified. We saw that the box topology on $\mathbb{R}^{\mathbb{N}}$ is quite poorly behaved, and the same is true in general. We will only mention it a few more times.

Exercise 7.4. Show that for an arbitrary product of topological spaces, $\mathcal{T}_{\text{prod}} \subseteq \mathcal{T}_{\text{box}}$, and that this containment is strict for infinite products.

Just as with $\mathbb{R}^{\mathbb{N}}$, the questions we should immediately seek to answer are of the form “Does an arbitrary product of topological spaces with property ϕ itself have property ϕ ?” This question needs to be a little more subtle now that we can consider uncountable products, so we actually have two definitions here.

Definition 7.5. A topological property ϕ is said to be countably productive if whenever $\mathcal{X} = \{(X_n, \mathcal{T}_n) : n \in \mathbb{N}\}$ is a collection of topological spaces each with property ϕ , then their product $X = \prod_{n \in \mathbb{N}} X_n$ with its product topology also has property ϕ .

Definition 7.6. A topological property ϕ is said to be productive (or sometimes arbitrarily productive) if whenever I is a nonempty indexing set and $\mathcal{X} = \{(X_\alpha, \mathcal{T}_\alpha) : \alpha \in I\}$ is a collection of topological spaces each with property ϕ , then their product $X = \prod_{\alpha \in I} X_\alpha$ with its product topology also has property ϕ .

Obviously every productive property is countably productive (and also finitely productive), but we will see below that the converse is not true. Most of these proofs will look similar to the proofs for $\mathbb{R}^{\mathbb{N}}$.

Proposition 7.7. Let I be a nonempty indexing set, and let $\mathcal{X} = \{(X_\alpha, \mathcal{T}_\alpha) : \alpha \in I\}$ be a collection of Hausdorff spaces. Then $X = \prod_{\alpha \in I} X_\alpha$ with the product topology is Hausdorff (and therefore also with the box topology).

In other words, the property of being Hausdorff is productive.

Proof. Exercise. This is essentially identical to the proof we gave for $\mathbb{R}^{\mathbb{N}}$. □

Proposition 7.8. Regularity is productive.

Proof. This proof is straightforward but tedious, so we omit it here and do not require that you complete it as an exercise (though you are welcome to do so if you like). □

Proposition 7.9. Normality is not countably productive.

Proof. We already know normality is not even finitely productive, since the $\mathbb{R}_{\text{Sorgenfrey}}$ is normal but $\mathbb{R}_{\text{Sorgenfrey}}^2$ is not. □

As you might predict, the product being countable or uncountable comes into play when dealing with countability properties. Countable products preserve all the countability properties except for ccc (which we have already seen behaves very weirdly with respect to products).

Proposition 7.10. *First countability, second countability, separability and metrizable (and complete metrizable) are all countably productive.*

Proof. For the duration of this proof, we let $\mathcal{X} = \{(X_n, \mathcal{T}_n) : n \in \mathbb{N}\}$ be a countable collection of topological spaces, and let $X = \prod_{n \in \mathbb{N}} X_n$ be their Cartesian product, with its product topology.

The proof for first countability is easy to understand but tedious to write down. Here is the idea. Suppose (X_n, \mathcal{T}_n) is first countable for every n , and let $x = (x_1, x_2, x_3, \dots) \in X$. We show there is a countable local basis at x .

For each $k \in \mathbb{N}$, let \mathcal{B}_{x_k} be a countable local basis at x_k , which we can find since (X_k, \mathcal{T}_k) is first countable. For each $n \in \mathbb{N}$, define:

$$L_n = \left\{ \prod_{k \in \mathbb{N}} U_k \in \mathcal{T}_{\text{prod}} : U_k \in \mathcal{B}_{x_k} \text{ for all } k \leq n \text{ and } U_k = X_k \text{ for all } k > n \right\}$$

That is, L_n is the collection of all basic open sets of the product topology such that the sets in the first n coordinates come from the countable local bases fixed earlier. Then L_n is countable, since it can be put into bijection with $\mathcal{B}_{x_1} \times \dots \times \mathcal{B}_{x_n}$ in the natural way. Finally, let

$$\mathcal{B}_x = \bigcup_{n \in \mathbb{N}} L_n$$

Then \mathcal{B}_x is countable (being a countable union of countable sets), and it is straightforward to show that \mathcal{B}_x is a local basis at x .

The proof for second countability is also tedious, but less so. Suppose (X_n, \mathcal{T}_n) is second countable for all n , and let \mathcal{B}_n be a countable basis for X_n . Just like in the definition of the product topology, let

$$\mathcal{S}_n = \{ \pi_n^{-1}(U) : U \in \mathcal{B}_n \}.$$

Then clearly \mathcal{S}_n is countable. Finally, let $\mathcal{S} = \bigcup_{n \in \mathbb{N}} \mathcal{S}_n$. Then \mathcal{S} is a countable subbasis on X , and the basis it generates is therefore also countable. This is the basis consisting of sets of the form $\prod_{n \in \mathbb{N}} U_n$, where $U_n = X_n$ for all but finitely many n , and finitely many remaining coordinates are basic open sets in the corresponding \mathcal{B}_i 's. It is easy to check that this basis generates the product topology.

The proof for separability is almost exactly the same as for Proposition 4.7, and so is left as an exercise for the reader.

The proof for metrizable is also almost exactly the same as for Proposition 4.8. Suppose (X_n, \mathcal{T}_n) is metrizable for all n . Let d_n be a metric on X_n that generates \mathcal{T}_n . Again, we may assume this metric is bounded by 1. Define the metric d_1 on X by

$$d_1(x, y) = \sup \left\{ \frac{d_k(x_k, y_k)}{k} : k \in \mathbb{N} \right\}.$$

This metric generates the product topology on X . The proof is essentially the same as for $\mathbb{R}^{\mathbb{N}}$, and is left as an exercise for the reader. This metric is also complete, provided all (X_k, d_k) 's are complete, and the proof is the same as it was for $\mathbb{R}^{\mathbb{N}}$. \square

These countability properties do not play well with larger products. For example:

Proposition 7.11. *Let I be an uncountable set. Then \mathbb{R}^I with its product topology is not first countable (and therefore not second countable or metrizable).*

In particular, first countability, second countability and metrizability are not productive.

Proof. We show this by finding a subset A of \mathbb{R}^I and a point $g \in \bar{A}$ such that no sequence from A converges to g .

First, define the set

$$A = \{ f \in \mathbb{R}^I : f(\alpha) = 1 \text{ for all but finitely many } \alpha \in I \}.$$

Then the “zero” point of \mathbb{R}^I (which formally speaking is the constant zero function $\bar{0} : I \rightarrow \mathbb{R}$) is in \bar{A} . To see this, let $U = \prod_{\alpha \in I} U_{\alpha}$ be a basic open set containing $\bar{0}$. Then there is a finite set $F \subseteq I$ such that $U_{\alpha} = \mathbb{R}$ for all $\alpha \in I \setminus F$. Define an element $f \in \mathbb{R}^I$ by

$$f(\alpha) = \begin{cases} 0 & \alpha \in F \\ 1 & \alpha \in I \setminus F \end{cases}$$

Then it is easy to see that $f \in U \cap A$, and since U was arbitrary this implies that $\bar{0} \in \bar{A}$.

However, no sequence from A converges to $\bar{0}$. To see this, let $\{f_n\}_{n \in \mathbb{N}}$ be any sequence in A . By definition of A , for each n there is a finite set $F_n \subseteq I$ such that $f_n(\alpha) \neq 1$ if and only if $\alpha \in F_n$. Let $C = \bigcup_{n \in \mathbb{N}} F_n$.

C is countable, being a countable union of finite sets. Since I is uncountable, this means $I \setminus C$ is uncountable and in particular is not empty, so let $\beta \in I \setminus C$. To be clear, this means that $f_n(\beta) = 1$ for all $n \in \mathbb{N}$. Finally, let $U = \prod_{\alpha \in I} U_{\alpha}$ be the basic open set such that $U_{\beta} = (-1, 1)$ and $U_{\alpha} = \mathbb{R}$ for all $\alpha \neq \beta$. Then clearly $\bar{0} \in U$, but U contains no element of the sequence $\{f_n\}$. \square

The question of separability is somewhat more subtle. The moral of that story is that separability is not productive, but that a “small” product of separable space is separable. The full extent of how large of a product you can take depends on the Continuum Hypothesis, as the general result says that a product of up to \mathfrak{c} -many separable spaces is separable, where \mathfrak{c} is the cardinality of \mathbb{R} . This is a consequence of the [Hewitt-Marczewski-Pondiczery theorem](#).

We summarize some results about productivity of topological properties below.

	finitely productive	countably productive	productive
Hausdorff	✓	✓	✓
regular	✓	✓	✓
normal	✗	✗	✗
first countable	✓	✓	✗
metrizable	✓	✓	✗
completely metrizable	✓	✓	✗
second countable	✓	✓	✗
separable	✓	✓	✗
ccc	independent of ZFC	independent of ZFC	independent of ZFC

After looking at this table, the first question that should occur to you is whether there are any topological properties that are finitely productive but not countably productive. The answer is of course yes, though none of the usual properties we have dealt with thus far fill this role. Here is a very easy exercise that very sharply illustrates the difference.

Exercise 7.12. Show that any finite product of discrete spaces with more than one point is discrete, but that any countably infinite product of discrete spaces each with more than one point is not discrete.

Finally, we mention the analogues to Propositions 4.13 and 4.14 in the context of arbitrary products. Just as with $\mathbb{R}^{\mathbb{N}}$, the product topology works exactly the way we would hope here. A sequence converges if and only if all of its component sequences converge, and a function to a product is continuous if and only if all of its component functions are continuous.

Proposition 7.13. *Let I be a nonempty indexing set, and let $\mathcal{X} = \{(X_\alpha, \mathcal{T}_\alpha) : \alpha \in I\}$ be a collection of nonempty topological spaces indexed by I . Let $X = \prod_{\alpha \in I} X_\alpha$ be the Cartesian product of the underlying sets in \mathcal{X} .*

Let $\{a_n\}_{n \in \mathbb{N}}$ be a sequence in X . Then the sequence converges to a point $x \in X$ if and only if the “coordinate sequence” $\{\pi_\alpha(a_n)\}_{n \in \mathbb{N}}$ in X_α converges to $\pi_\alpha(x)$ for all $\alpha \in I$.

Proof. Exercise. □

Proposition 7.14. *Let (Y, \mathcal{U}) be a topological space, and let I , \mathcal{X} and X be as in the previous proposition. Let $f : Y \rightarrow X$ be a function, where X has the product topology. Then f is continuous if and only if $\pi_\alpha \circ f : Y \rightarrow X_\alpha$ is continuous for every $\alpha \in I$.*

Proof. Exercise. □

8 A final note on completeness, and completions

This section should be thought of as supplementary, though I do think you will find it interesting.

Earlier we defined the uniform metric d_u on $\mathbb{R}^{\mathbb{N}}$, and proved that the metric space $(\mathbb{R}^{\mathbb{N}}, d_u)$ is complete. The usefulness of this metric is that one can always define it for any product of metric spaces, and the resulting metric on a product of complete metric spaces is always complete.

Proposition 8.1. *Let I be a nonempty indexing set, and let $\mathcal{X} = \{(X_\alpha, d_\alpha) : \alpha \in I\}$ be a collection of complete metric spaces. Without loss of generality, we may assume that each d_α is bounded by 1. Let $X = \prod_{\alpha \in I} X_\alpha$ be the product of the underlying sets in \mathcal{X} , and define the uniform metric d_u on X by*

$$d_u(x, y) = \sup \{ d_\alpha(\pi_\alpha(x), \pi_\alpha(y)) : \alpha \in I \}$$

Then (X, d_u) is a complete metric space.

Proof. The proof is essentially identical to the one given for the uniform metric on $\mathbb{R}^{\mathbb{N}}$. \square

Unfortunately, we know that a product of completely metrizable spaces with its product topology need not be completely metrizable, since we know it need not even be metrizable.

The final question we address here is the difference between metrizability and complete metrizability. We already know that \mathbb{Q} is a space that is metrizable but not completely metrizable. However, all is not lost. We know that \mathbb{Q} can be embedded—isometrically embedded, no less—as a subspace of \mathbb{R} , which *is* complete. Formally, we refer to \mathbb{R} as the completion of \mathbb{Q} , in the sense that it is the smallest complete metric space into which \mathbb{Q} isometrically embeds. We will formalize what we mean by “smallest” later.

For now, we prove that any metric space can be embedded into a complete metric space in this same sense.

Definition 8.2. *Let X be a set, and let (Y, d) be a metric space. Let*

$$B(X, Y) := \{ f : X \rightarrow Y : f \text{ is bounded} \}.$$

(where a function f is called bounded if its range $f(X)$ is a bounded subset of Y , according to its metric). Define a metric ρ on $B(X, Y)$ by $\rho(f, g) = \sup \{ d(f(x), g(x)) : x \in X \}$. This is called the sup metric.

Note that the sup metric is well-defined since for any $f, g \in B(X, Y)$, the set $f(X) \cup g(X)$ is bounded. It is an elementary exercise to show that ρ is actually a metric. Also note that the set $B(X, Y)$ is just a subset of Y^X , the set of all functions Y to X . Since Y is a metric space, we can think of Y^X as a Cartesian product $\prod_{x \in X} Y$ of metric spaces, and in that sense the metric ρ we defined is precisely the uniform metric on Y^X , except that it can take on infinite values. The set $B(X, Y)$ is just the subset of this metric space on which the metric always takes on finite values.

Proposition 8.3. *Let X be a set, and let (Y, d) be a metric space. Let ρ be the sup metric on $B(X, Y)$. Then $(B(X, Y), \rho)$ is a complete metric space.*

Proof. This proof is essentially identical to the proof of Proposition 5.5. The metric d_u we defined in that proof is the sup metric for the set $B(\mathbb{N}, \mathbb{R})$, where \mathbb{R} has its usual bounded metric \bar{d} . The details are left to the reader. \square

Here is the payoff of all of this. This result is a preview of some of things we will study later in the course. The great usefulness of being able to define topologies on large products and function spaces is that we can embed spaces into them relatively easily. This is our first example of that.

Theorem 8.4. *Let (X, d) be a metric space. Then there is an isometric embedding of X into a complete metric space.*

Proof. The complete metric space into which we will embed X is $(B(X, \mathbb{R}), \rho)$, where ρ is the sup metric defined just above, thought of as a topological space with the metric topology generated by ρ .

Fix an arbitrary point $x_0 \in X$. For any point $a \in X$, define $f_a : X \rightarrow \mathbb{R}$ by

$$f_a(x) = d(x, a) - d(x, x_0).$$

First, we show that f_a is bounded for all $a \in X$. Given any $x \in X$, by the triangle inequality we have:

$$d(x, a) \leq d(x, x_0) + d(a, x_0) \quad \text{and} \quad d(x, x_0) \leq d(x, a) + d(a, x_0)$$

These two inequalities respectively rearrange to

$$d(x, a) - d(x, x_0) \leq d(a, x_0) \quad \text{and} \quad -d(a, x_0) \leq d(x, a) - d(x, x_0)$$

from which it follows that

$$|f_a(x)| = |d(x, a) - d(x, x_0)| \leq d(a, x_0).$$

The quantity $d(a, x_0)$ does not depend on x , so this shows that $f_a \in B(X, \mathbb{R})$. This allows us to define our embedding of X . Let $\Phi : X \rightarrow B(X, \mathbb{R})$ be defined by $\Phi(a) = f_a$. It remains to show that Φ is an isometry, which is to say that

$$\rho(\Phi(a), \Phi(b)) = \rho(f_a, f_b) = d(a, b)$$

for all $a, b \in X$. Fix $a, b \in X$. Then by definition of Φ , we have

$$\begin{aligned} \rho(\Phi(a), \Phi(b)) &= \rho(f_a, f_b) \\ &= \sup \{ |f_a(x) - f_b(x)| : x \in X \} \\ &= \sup \{ |d(x, a) - d(x, x_0) - d(x, b) + d(x, x_0)| : x \in X \} \\ &= \sup \{ |d(x, a) - d(x, b)| : x \in X \} \\ &\leq d(a, b) \end{aligned}$$

where the last line is by the reverse triangle inequality. When $x = a$, we have:

$$|d(x, a) - d(x, b)| = |d(a, a) - d(a, b)| = d(a, b),$$

and therefore the inequality at the end of our derivation must actually be an equality. This completes the proof. \square

Proposition 8.5. *Let (X, d) be a complete metric space, and let $C \subseteq X$ be a nonempty closed set (in the topology generated by d , of course). Then C , with metric defined by restricting d to C , is a complete metric space.*

Proof. Let $\{x_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in C . Then it converges to some $x \in X$ since X is complete. But C is closed, and therefore $x \in C$. \square

This allows us to define what we promised.

Definition 8.6. *Let (X, d) be a metric space. Let $f : X \rightarrow Y$ be any isometric embedding of X into a complete metric space (Y, d') . Then the subspace $\overline{f(X)}$ of Y is a complete metric space, and is called the completion of X .*

The theorem above tells us that X actually can be isometrically embedded in at least one complete metric space, so this definition makes sense. What is less clear is that we are justified in saying “*the* completion” rather than “*a* completion” since *a priori* the space $\overline{f(X)}$ might be different for different embeddings into different spaces. However, it is not hard to check that any two spaces obtained in this way are globally isomorphic, as we would hope. We therefore usually say that the completion of a metric space is “complete up to isometry”.

To get an idea for why this is true, let (X, d) be a metric space as in the definition, and let $f : X \rightarrow Y$ and $g : X \rightarrow Z$ be isometric embeddings of X into complete metric spaces (Y, d_Y) and (Z, d_Z) . Then the function

$$g \circ f^{-1} : f(X) \rightarrow g(X)$$

is an isometry since g and f are both isometries. It is left as an exercise for the reader to check that this function can be uniquely extended to an isometry $\overline{f(X)} \rightarrow \overline{g(X)}$.