Stronger separation axioms

1 Motivation

While studying sequence convergence, we isolated three properties of topological spaces that are called separation axioms or T-axioms. These were called T_0 (or Kolmogorov), T_1 (or Frećhet), and T_2 (or Hausdorff). To remind you, here are their definitions:

Definition 1.1. *A topological space $(X, \mathcal{T})$ is said to be T_0 (or much less commonly said to be a Kolmogorov space), if for any pair of distinct points $x, y \in X$ there is an open set U that contains one of them and not the other.*

Recall that this property is not very useful. Every space we study in any depth, with the exception of indiscrete spaces, is T_0.

Definition 1.2. *A topological space $(X, \mathcal{T})$ is said to be T_1 if for any pair of distinct points $x, y \in X$, there exist open sets U and V such that U contains x but not y, and V contains y but not x.*

Recall that an equivalent definition of a T_1 space is one in which all singletons are closed. In a space with this property, constant sequences converge only to their constant values (which need not be true in a space that is not T_1—this property is actually equivalent to being T_1).

Definition 1.3. *A topological space $(X, \mathcal{T})$ is said to be T_2, or more commonly said to be a Hausdorff space, if for every pair of distinct points $x, y \in X$, there exist* disjoint *open sets U and V such that $x \in U$ and $y \in V$.*

This property is a very important one. It implies for example that convergent sequences have unique limit points. It is so fundamental to the way we think about topological spaces that when the abstract concept of a topological space was first formalized (by Felix Hausdorff), topological spaces were *defined* to have this property.

All three of these properties are called "separation axioms" because they codify the extent to which a topology can distinguish between points in the underlying set, or in other words how well it can separate points with open sets. Recall that you also proved on a (one-star) Big List problem that $T_2 \Rightarrow T_1 \Rightarrow T_0$, which should feel satisfying.

In this section we will define and study two stronger separation properties, and perhaps some variations on them. Since Hausdorff spaces can already separate points with open sets, both of our properties will involve separating more complicated things with open sets.

A warning before proceeding. The names of these two properties and their corresponding T-numbers are not precisely agreed upon in the mathematical literature. Traditionally they are defined in a way which makes very little sense to me, and more modern works will usually define them the way I am defining them.

In particular, the definitions of "regular", "T_3", "normal", and "T_4" given below disagree at least on some level with Munkres' textbook and with *Counterexamples in Topology.* They will agree, for example, with Wikipedia. So be sure to read the definitions carefully. I will address these discrepancies and explain my choices as they arise in the sections below.

2 Regularity and the T_3 axiom

First, we define a property that lets us separate points from closed sets. To reiterate my warning from above, please take care to read exactly what I am defining here and note that other definitions you see (such as in Munkres' text) may differ slightly, in ways I will explain below.

Definition 2.1. *A topological space $(X, \mathcal{T})$ is said to be regular if for any $x \in X$ and any closed set C not containing x, there are disjoint open sets U and V such that $x \in U$ and $C \subseteq V$.*

Instead of separating points from points (as in a Hausdorff space), we can separate points from closed sets. Many spaces we have studied so far are regular.

Example 2.2. The following topological spaces are all regular.

1. Any discrete space is trivially regular.

2. $\mathbb{R}^n_{\text{usual}}$ is regular. To see this, let $x \in \mathbb{R}^n$ and let $C \subseteq \mathbb{R}^n$ be closed and not contain x. Since $x \in \mathbb{R}^n \setminus C$ which is open, there is an $\epsilon > 0$ such that $x \in B_\epsilon(x) \subseteq \mathbb{R}^n \setminus C$.

   Now let
   $$U := \bigcup_{y \in C} B_{\epsilon/2}(y).$$

   Then U is an open set containing C, and $U \cap B_{\epsilon/2}(x) = \emptyset$ (check this!).

   Notice that all we used about the reals here is that the topology has a basis of ϵ-balls. Later this will tell us that any metric space—any space whose topology has a basis of ϵ-balls, essentially—is regular.

3. $\mathbb{R}_{\text{Sorgenfrey}}$ is regular, as we will see shortly.

4. $\mathbb{R}_{\text{co-countable}}$ is not regular, since any two nonempty open sets intersect.

5. (Prepare to be annoyed.) $(X, \mathcal{T}_{\text{indiscrete}})$ is vacuously regular since the only nonempty closed set is X, which contains every point.

This last example is just awful. Regularity is supposed to be a separation axiom that says you can do *even better than* separating points, and yet the indiscrete topology is regular despite being unable to separate anything from anything else.

This means that the definition we gave above should not be the correct one; it is the morally correct one, but not the technically correct one. What we actually want is for the property of being able to separate points from closed sets to be *stronger* than the property of being able to separate points from other points. The problem, which should now be apparent, is that points may not be closed. This brings us to:

Definition 2.3. *A topological space $(X, \mathcal{T})$ is said to be $\underline{T_3}$ if it is both T_1 and regular.*

Under this improved definition, it is clear that $T_3 \Rightarrow T_2$, since you can separate two points x and y by separating x and $\{y\}$, the latter of which is always closed in a T_1 space.

Example 2.4. $\mathbb{R}^n_{\text{usual}}$, $\mathbb{R}_{\text{Sorgenfrey}}$, and any discrete space are all T_3. An indiscrete space with more than one point is regular but not T_3.

Note that we only actually need T_0 in the definition above, but in this context T_0, T_1 and T_2 are all equivalent, as the following proposition shows. We chose to use T_1 in the definition above because it *sounds* like the property we need, so it is easiest to understand and remember in that role.

Proposition 2.5. *Let $(X, \mathcal{T})$ be regular. Then $(X, \mathcal{T})$ is T_0 if and only if it is T_1 if and only if it is T_2.*

Proof. We already know that in any space, $T_2 \Rightarrow T_1 \Rightarrow T_0$. So assume X is regular and T_0, and let $x \neq y \in X$. Then by T_0-ness, without loss of generality there exists an open set U containing x but not y. But then $X \setminus U$ is a closed set that contains y but not x, so by regularity there are disjoint open sets V_1 and V_2 containing C (and therefore y) and x, respectively. Therefore X is T_2 (and therefore T_1). $\square$

With that technicality out of the way, the first result we will see is a useful alternative characterization of regularity, whose proof is a routine exercise. This is a property that is easy to take for granted in a space like the reals. If you did the Big List exercise in which you were asked to prove a version of the Baire Category Theorem, you likely already came across a use of this property.

Proposition 2.6. *A topological space $(X, \mathcal{T})$ is regular if and only if for every point $x \in X$ and every open set U containing x, there is an open set V such that $x \in V \subseteq \overline{V} \subseteq U$.*

Proof. **Exercise.** (This is very straightforward. You should draw yourself a picture first, convince yourself that it is obvious from the picture, then write down the proof that the picture tells you to write down.) $\square$

Next, we will see that the T_3 property is actually strictly stronger than the Hausdorff property. We already know that $T_3 \Rightarrow T_2$, but we do not yet know that T_3 is strictly stronger.

Example 2.7. Let $K = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \subseteq \mathbb{R}$. Define a collection of subsets of $\mathbb{R}$ as follows:

$$\mathcal{B} = \{ (a,b) : a < b \in \mathbb{R} \} \cup \{ (a,b) \setminus K : a < b \in \mathbb{R} \}.$$

Then the set $\mathcal{B}$ is a basis on $\mathbb{R}$. We already know that the open intervals are closed under finite intersections, and the sets of the form $(a,b) \setminus K$ are also clearly closed under intersections, so it is not hard to see that $\mathcal{B}$ is a basis. The topology that $\mathcal{B}$ generates is called the <u>K topology</u>, and we denote $\mathbb{R}$ with this topology by $\mathbb{R}_K$. Note that the K topology strictly refines the usual topology (since every basic open set of the usual topology is open in the K topology). In particular this implies that $\mathbb{R}_K$ is Hausdorff, since every pair of sets that witnesses that property in $\mathbb{R}_{\text{usual}}$ is also open in $\mathbb{R}_K$.

On the other hand, $\mathbb{R}_K$ is not regular, and therefore not T_3. To see this, note that K is a closed set in $\mathbb{R}_K$, but the point 0 and the closed set K cannot be separated by disjoint open sets. Clearly any set of the form (a,b) containing 0 will intersect K, and therefore if $U = (a,b) \setminus K$ is a basic open set containing 0, any open interval around an element of $K \cap (a,b)$ will intersect U.

With some examples and relationships to other properties out of the way, we can now examine the T_3 property itself, and see how it interacts with products, subspaces, etc.

Proposition 2.8. *Regularity (and therefore T_3) is a topological invariant.*

Proof. **Exercise.** □

Proposition 2.9. *Regularity is a hereditary property. Since we already know T_1 is hereditary, this also means T_3 is hereditary.*

Proof. Suppose $(X, \mathcal{T})$ is regular and $A \subseteq X$ is a subspace. Let $x \in A$ and let $C \subseteq A$ be a closed subset not containing x. By definition of the subspace topology there is a closed subset $D \subseteq X$ such that $C = D \cap A$. Then $x \notin D$, and since X is regular we can find disjoint open subsets U_1 and U_2 of X containing D and x respectively. Then $U_1 \cap A$ and $U_2 \cap A$ are disjoint open subsets of A containing C and x, respectively. □

Proposition 2.10. *Regularity is finitely productive. Since we already know T_1 is finitely productive, this also means T_3 is finitely productive.*

Proof. **Exercise.** (Hint: This is very straightforward if you use the characterization of regularity given in Proposition 2.6.) □

One final nice fact about regularity, that has a particularly nice proof.

Proposition 2.11. *Let $(X, \mathcal{T})$ be a topological space that has a basis of clopen sets. Then $(X, \mathcal{T})$ is regular.*

Proof. Suppose $\mathcal{B}$ is a basis for $(X, \mathcal{T})$ consisting entirely of clopen sets, fix a point $x \in X$ and a closed set $C \subseteq X$ not containing x. Then $x \in X \setminus C$ which is open, and therefore there is a basic clopen set $B \in \mathcal{B}$ such that $x \in B \subseteq X \setminus C$.

But then $C \subseteq X \setminus B$, and therefore B and $X \setminus B$ are the desired open sets separating x and C. $\square$

Corollary 2.12. *The Sorgenfrey Line (and therefore all of its finite powers and all of their subspaces) is regular.*

Proof. As you proved on a Big List problem, every basic open set $[a, b)$, $a < b \in \mathbb{R}$ is also closed. $\square$

WARNING: There is no consensus in mathematics about which of the two properties defined in this section should be called "regular" and which one should be called "T_3". Even from a historical standpoint, both ways of naming them have merit (you can read about this in this Wikipedia article on the history of separation axioms).

In particular, note that the choices made here disagree with Munkres's text. That book defines "regular" to mean what we call "T_3", and only mentions the T-axioms in a note in which he says T_3 and "regular" mean the same thing. Munkres does not give a name to the property we call "regular". *Counterexamples in Topology*, on the other hand, defines "regular" and "T_3" in exactly the reverse way to how we define them. Finally, if you look things up on Wikipedia, they agree with our definitions (they go out of their way to call T_3 spaces "regular Hausdorff" spaces sometimes, but usually just give both names).

The motivation for the choice of names in this text is simply that the T-axioms should imply each other. We will shortly have T_4 as well, and it seems correct for this to be true:

$$T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0.$$

No one disagrees on the definitions of T_0, T_1, and T_2/Hausdorff, but using the definitions in *Counterexamples in Topology*, T_4 would not imply T_3, and neither of them would imply T_2.

So again, please accept my apology on behalf of the mathematicians of the 20th century. We dropped the ball on this one. I am doing what I think is right. What you need to take away from this is that you need to *pay attention* when you read other sources. Be sure to find out (or try to deduce) exactly what they mean when they use these terms.

3 Normality and the T_4 axiom

At this point, this property should seem like the natural next step.

Definition 3.1. *A topological space $(X, \mathcal{T})$ is said to be normal if for any two disjoint, nonempty, closed subsets $C, D \subseteq X$, there are disjoint open sets U and V containing C and D, respectively.*

Example 3.2.

1. Any discrete space is trivially normal.

2. $\mathbb{R}^n_{\text{usual}}$ is normal. To see this, let $C, D \subseteq \mathbb{R}^n$ be closed. For each $x \in C$, find an $\epsilon_x > 0$ such that $\mathcal{B}_{\epsilon_x}(x) \subseteq \mathbb{R}^n \setminus D$, which we can do since $\mathbb{R}^n \setminus D$ is open. Similarly, for each $x \in D$, find $\delta_x > 0$ such that $B_{\delta_x}(x) \subseteq \mathbb{R}^n \setminus C$.

   Then we have two open sets

   $$U := \bigcup_{x \in C} B_{\epsilon_x}(x) \quad \text{and} \quad V = \bigcup_{x \in D} B_{\delta_x}(x)$$

   which contain C and D, respectively. These sets might intersect, however, so instead let

   $$U' := \bigcup_{x \in C} B_{\epsilon_x/2}(x) \quad \text{and} \quad V' = \bigcup_{x \in D} B_{\delta_x/2}(x).$$

   These open sets must be disjoint (check this!) and contain C and D, respectively. Therefore $\mathbb{R}^n_{\text{usual}}$ is normal.

3. The Sorgenfrey Line is normal.

4. $\mathbb{R}_{\text{co-countable}}$ is not normal, again since any two open sets intersect.

5. (Prepare to be annoyed again.) Any indiscrete space is vacuously normal, since there are no pairs of disjoint closed nonempty subsets.

So again we run into this silly issue that indiscrete spaces are normal. We also run into the issue that normality does not imply regularity or Hausdorff, since singletons need not be closed. So again, the fix for this is to define the next T-axiom by including T_1-ness as a condition.

Definition 3.3. *A topological space $(X, \mathcal{T})$ is said to be $\underline{T_4}$ if it is both T_1 and normal.*

(Again, different texts and sources will disagree with me and with each other on this in the same way they disagree about regular/T_3.)

With this definition in hand, it is easy to see that $T_4 \Rightarrow T_3$, since separating a point and a closed set is equivalent to separating the corresponding singleton from the closed set.

Just as with regularity, there is an alternative characterization that is useful and worth pointing out.

Proposition 3.4. *A topological space $(X, \mathcal{T})$ is normal if and only if for every open set U and every closed set $C \subseteq U$, there is an open set V such that $C \subseteq V \subseteq \overline{V} \subseteq U$.*

Proof. **Exercise.** (Again, this is very straightforward, and should not require any creativity. Make sure to draw yourself a picture first.) $\square$

Just as we did with regularity and T_3, we can now examine the T_4 property itself and see how it interacts with products, subspaces, etc.

Proposition 3.5. *Normality (and therefore T_4) is a topological invariant.*

Proof. **Exercise.** (Nothing tricky to do here.) $\square$

However, normality does not play as nicely as regularity.

Proposition 3.6. *Normality is* <u>*not*</u> *hereditary.*

Proof. We show that any non-normal topological is a subspace of a normal space. This proof isn't very satisfying, because the normal space in which our given non-normal space is a subspace is vacuously normal.

Let $(Y,\mathcal{U})$ be any non-normal topological space, and let ∞ be a symbol that is not an element of Y. The underlying set for the new space we are constructing will be $X := Y \cup \{\infty\}$. We define a topology $\mathcal{T}$ on X by $\mathcal{T} = \mathcal{U} \cup \{X\}$. Take a moment to convince yourself that $\mathcal{T}$ is indeed a topology on X.

First, note that the subspace topology on Y in this space just *equals* $\mathcal{U}$, so our original non-normal space $(Y,\mathcal{U})$ is a subspace of $(X,\mathcal{T})$.

To see that $(X,\mathcal{T})$ is normal, note that the only open set in this space that contains the new point ∞ is X itself. In other words, every non-empty closed set in this space contains ∞. This means that *there are no* pairs of disjoint nonempty closed sets, and so the space is vacuously normal. $\square$

Coming up with an *interesting* example of a normal space with non-normal (abnormal?) subspace is actually quite tricky. A nice example you can look up is in *Counterexamples in Topology* (Example 86), though you probably will not get much out of it at the moment. Wait until after we have learned about ω_1...

What *is* worth doing though is figuring out why the "obvious proof" that normally should be hereditary fails.

Let $(X,\mathcal{T})$ be normal and let $A \subseteq X$ be a subspace. Let $C_1, C_2 \subseteq A$ be disjoint, nonempty closed subsets in the subspace topology. Then there exist two sets D_1 and D_2 which are closed in X such that $C_i = A \cap D_i$ for $i = 1, 2$. We want to use the normality of X to separate D_1 and D_2, but here is where it breaks down. Can you see why?

All is not lost, however. Just like how we showed separability was not hereditary but on the Big List you showed that an *open* subspace of a separable space is separable, there is a similar patch here.

Proposition 3.7. *Every* <u>*closed*</u> *subspace of a normal space is normal.*

Proof. **Exercise.** (Pay particular attention to why the problem you found with the "naive" proof is no longer a problem in this case.) $\square$

Okay, so normality is one of the rare (at least so far) non-hereditary properties. Even more unusual is the following:

Proposition 3.8. *Normality is not finitely productive.*

Proof. We show this by showing that the product of the Sorgenfrey Line—which we already noted is normal—with itself is not normal. This space, often called the Sorgenfrey Square or similar, is quite an interesting one.

So let $(\mathbb{R}, \mathcal{S})$ be the Sorgenfrey line with its usual basis, and consider $(\mathbb{R}^2, \mathcal{T})$ where $\mathcal{T}$ is the product topology generated by basis elements that look like $[a, b) \times [c, d)$, in the usual way. We show that $(\mathbb{R}^2, \mathcal{T})$ is not normal.

Let $A \subseteq \mathbb{R}^2$ be the antidiagonal of the plane. That is, the graph of the function $y = -x$, or in other words $A = \{ (x, -x) : x \in \mathbb{R} \}$. Take a moment to convince yourself that A is a closed, discrete subspace of $(\mathbb{R}^2, \mathcal{T})$. Now split A into two subsets:

$$A_1 = \{ (x, -x) : x \in \mathbb{Q} \} \quad \text{and} \quad A_2 = \{ (x, -x) : x \notin \mathbb{Q} \}.$$

Then A_1 and A_2 are closed, disjoint subspaces of $(\mathbb{R}^2, \mathcal{T})$, but they cannot be separated by open subsets of $(\mathbb{R}^2, \mathcal{T})$. (Actually showing that these two sets cannot be separated by open sets is somewhat tricky. I invite you to read the proof in Munkres' *Topology*, as Example 3 on page 198. It is quite a novel argument, that assumes $(\mathbb{R}, \mathcal{T})$ is normal and uses this to define an injection from the power set of A to A, which is impossible.) $\square$

The previous example also serves another purpose, which is to distinguish regularity from normality. Normality certainly *seems* stronger, but this is the first example we have of a space that fails to be normal for a non-trivial reason. To be clear, we have shown that $(\mathbb{R}^2, \mathcal{T})$ is regular (since the Sorgenfrey Line is regular and regularity is finitely productive) but not normal. This demonstrates that normality is a strictly stronger property than regularity.

4 One last result

We end with a nice result involving normality. This sort of theorem, about how some combination of topological properties implies some other topology property, is the bread and butter of point-set topology.

This result should seem somewhat odd, in the sense that it is not obvious why the properties should interact this way, but the proof is very constructive and easy to follow.

Theorem 4.1. *Every regular, second countable topological space is normal.*

Proof. Suppose $(X, \mathcal{T})$ is regular and second countable with $\mathcal{B}$ a countable basis for $\mathcal{T}$, and let C and D be disjoint, nonempty, closed subsets of X. We wish to separate these two closed sets with open sets.

For each point x in C, by regularity (specifically the characterization from Proposition 2.6) we can find a basic open set $U_x \in \mathcal{B}$ that contains x and such that $\overline{U_x} \cap D = \emptyset$. Similarly for each $y \in D$ find an open set $V_y \in \mathcal{B}$ that contains y and $\overline{V_y} \cap C = \emptyset$.

Then we have:

$$C \subseteq \bigcup_{x \in C} U_x \quad \text{and} \quad D \subseteq \bigcup_{y \in D} V_y.$$

However, we took care to pick each of these U_x's and V_y's from the countable basis $\mathcal{B}$, so each of these unions is really a countable union. Re-indexing the sets, we can write:

$$C \subseteq \bigcup_{x \in C} U_x = \bigcup_{n \in \mathbb{N}} U_n \quad \text{and} \quad D \subseteq \bigcup_{y \in D} V_y = \bigcup_{n \in \mathbb{N}} V_n.$$

It seems as though these two unions are the open sets that work for us, but they might intersect. This is easily fixed, however. For each $n \in \mathbb{N}$, let

$$U_n' = U_n \setminus \left(\bigcup_{k=1}^{n} \overline{V_k} \right) \quad \text{and} \quad V_n' = V_n \setminus \left(\bigcup_{k=1}^{n} \overline{U_k} \right)$$

Each of these newly defined sets is open since we are taking an open set and removing a closed set. This is the point at which second countability is doing something for us that we could not have done otherwise. The ability to write each of the unions earlier as countable unions allows us to only have union together *finitely* many closed sets at this stage, which is crucial.

By construction, none of the $\overline{V_k}$'s intersect C and none of the $\overline{U_k}$'s intersect D, and therefore we still have that

$$C \subseteq U := \bigcup_{n \in \mathbb{N}} U_n' \quad \text{and} \quad D \subseteq V := \bigcup_{n \in \mathbb{N}} V_n'.$$

These two open sets U and V, finally, are disjoint. To see this, suppose $x \in U$. Then $x \in U_k'$ for some $k \in \mathbb{N}$. By construction of the U_n''s, this means $x \notin V_i'$ for all $i = 1, \ldots, k$. On the other hand if $i > k$, $V_i' \cap \overline{U_k} = \emptyset$, so $x \notin V_i'$. Therefore $x \notin V$, and so $U \cap V = \emptyset$, as required.

□