GRT Learning Seminar 05/02/2016 ①

Quantum Symmetric Pairs

(Peter Crooks)

① Classical Setup

Notation	Example
• $\mathfrak{g}$ f.d.s.s. $/\mathbb{C}$	$\mathfrak{sl}_n(\mathbb{C})$
• $\theta \in \mathrm{Aut}_{\mathbb{C}}(\mathfrak{g})$ involution	$\theta(X) = -X^T$
• $\mathfrak{g}^\theta = \{\xi \in \mathfrak{g} : \theta(\xi) = \xi\}$	$\mathfrak{sl}_n(\mathbb{C})^\theta = \mathfrak{so}_n(\mathbb{C})$
• G conn. s.c. w/ Lie alg. $\mathfrak{g}$; $K \subseteq G$ conn., closed w/ Lie alg. $\mathfrak{g}^\theta$	$SO_n(\mathbb{C}) \subseteq SL_n(\mathbb{C})$
• G/K "symmetric space"	$SL_n(\mathbb{C})/SO_n(\mathbb{C}) \underset{\uparrow}{\cong} \{A \in M_{n\times n}(\mathbb{C}) : A^T = A,\ \det A = 1\}$ as varieties

② Objective

$(\mathfrak{g}, \mathfrak{g}^\theta) \rightsquigarrow U(\mathfrak{g}^\theta) \subseteq U(\mathfrak{g})$

"symmetric pair"

$B \subseteq U_q(\mathfrak{g}) =: \mathcal{U}$

"quantum analogue of $U(\mathfrak{g}^\theta)$"

Note: $U_q(\mathfrak{g}^\theta) \not\subseteq U_q(\mathfrak{g})$

Desired Properties of B

- B is a left coideal subalgebra of $\mathcal{U}$.
- B specializes to $U(\mathfrak{g}^\theta)$ as $q \to 1$.
- B gives rise to "quantum symmetric spaces"

③ Construction of B [Letzter]

(a) More Notation

- $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ triangular decomposition

$$\underbrace{\Delta_-}_{\mathfrak{n}_-} \cup \underbrace{\Delta_+}_{\mathfrak{n}_+} = \Delta$$

- $\Pi = \{\alpha_1, \dots, \alpha_n\} \rightsquigarrow e_i, f_i, h_i \in \mathfrak{g}$

  simple roots $\qquad x_i, y_i, t_i^{\pm 1} \in \mathcal{U}$

- T = free abelian group generated by $\{t_i^{\pm 1}\}$
- Q = root lattice
- <u>Note:</u> $\tau : Q \xrightarrow{\cong} T$

  $\alpha_i \longmapsto t_i$

(b) Quantum Lift $\tilde{\theta}$

<u>Assumption:</u> θ is maximally split w.r.t. $\mathfrak{h}$, i.e.:

(i) $\theta(\mathfrak{h}) = \mathfrak{h}$

(ii) If $\theta(h_i) = h_i$, then $\theta(e_i) = e_i$, $\theta(f_i) = f_i$

(iii) If $\theta(h_i) \neq h_i$, then $\theta(e_i) \in \mathfrak{g}_\alpha$ for some $\alpha \in \Delta_-$ and $\theta(f_i) \in \mathfrak{g}_\beta$ for some $\beta \in \Delta_+$.

(iv) $\mathfrak{h}$ is maximally split w.r.t. θ.

<u>Note:</u> (ii), (iii) $\Rightarrow$ θ induces a root system autom. Θ

$$\Delta_\Theta = \{\alpha \in \Delta : \Theta(\alpha) = \alpha\}$$

$$\Pi_\Theta = \Pi \cap \Delta_\Theta$$

<u>Note:</u> $\theta(h_i) = h_i \Leftrightarrow \Theta(\alpha_i) = \alpha_i \Leftrightarrow \alpha_i \in \Pi_\Theta$.

<u>Lemma:</u> $\exists$ a permutation p of $\{i : \alpha_i \in \Pi - \Pi_\Theta\}$ such that $\alpha_i \in \Pi - \Pi_\Theta \Rightarrow \Theta(-\alpha_i) - \alpha_{p(i)}$ is a non-negative integral combination of the roots in Π_Θ.

$\Pi^* :=$ maximal subset of $\Pi - \Pi_{\Theta}$ such that

$i \neq p(i) \Rightarrow$ exactly one of $\alpha_i, \alpha_{p(i)}$ is in Π^*.

$i = p(i) \Rightarrow \alpha_i \in \Pi^*$.

Properties of (i)–(iii) restated

(i) $\theta(\mathfrak{h}) = \mathfrak{h}$

(ii) If $\alpha_i \in \Pi_{\Theta}$, then $\theta(e_i) = e_i$, $\theta(f_i) = f_i$

(iii) If $\alpha_i \in \Pi^*$, then:

$$\theta(f_i) = (\operatorname{ad} e_{i_1}^{(m_1)} e_{i_2}^{(m_2)} \cdots e_{i_r}^{(m_r)})(e_{p(i)})$$

$$\theta(f_{p(i)}) = (-1)^{m_1 + \dots + m_r} (\operatorname{ad} e_{i_r}^{(m_r)} e_{i_{r-1}}^{(m_{r-1})} \cdots e_{i_1}^{(m_1)})(e_i)$$

$$e_{i_j}^{(m_j)} = \frac{e^{m_j}}{m_j!} \qquad \alpha_{i_1}, \dots, \alpha_{i_r} \in \Pi_{\Theta}$$

Theorem There exists a $\mathbb{C}$-alg. automorphism $\tilde{\theta} : U \to U$ such that:

(i) $\tilde{\theta}(\tau(\lambda)) = \tau(\Theta(-\lambda))$ (so $\tilde{\theta}(T) = T$)

(ii) If $\alpha_i \in \Pi_{\Theta}$, then $\tilde{\theta}(x_i) = x_i$ $\tilde{\theta}(y_i) = y_i$

(iii) If $\alpha_i \in \Pi^*$, then

$$\tilde{\theta}(y_i) = (\operatorname{ad}_r x_{i_1}^{(m_1)} x_{i_2}^{(m_2)} \cdots x_{i_r}^{(m_r)})(t_{p(i)}^{-1} x_{p(i)})$$

$$\tilde{\theta}(y_{p(i)}) = (-1)^{m_1 + \dots + m_r} (\operatorname{ad} x_{i_r}^{(m_r)} \cdots x_{i_1}^{(m_1)})(t_i^{-1} x_i)$$

(iv) $\tilde{\theta}(q) = q^{-1}$.

Remark To define $\tilde{\theta}$ on $x_i, x_{p(i)}$ ($\alpha_i \in \Pi^*$), use Lusztig's automorphism T_{w_0}.

w_0 ← longest element of W_{Θ}; W_{Θ} = Weyl group of the root system for Δ_{Θ}

Remark $\tilde{\theta}$ is not an involution of U.

(it is an involution on T).

(c) Definition of B

Generators of $U(\mathfrak{g}^\theta)$	Generators of B
• $e_i, h_i, f_i \ldots \alpha_i \in \Pi_\Theta$	• $x_i, t_i^{\pm 1}, y_i \quad \alpha_i \in \Pi_\Theta$
• $\mathfrak{h}^\theta$	• $T_\Theta = \{\tau(\lambda) : \lambda \in Q,\ \Theta(\lambda) = \lambda\}$
	• $B_i = y_i t_i + \tilde{\theta}(y_i) t_i \quad \alpha_i \in \Pi - \Pi_\Theta$

$B :=$ subalgebra of U generated by these elements.

Theorem:

(i) B is a left coideal subalg. of U.

(ii) B specializes to $U(\mathfrak{g}^\theta)$ as $q \to 1$.

(iii) If $B \subseteq C \subseteq U$, then $B = C$. (maximality properties)

C ← subalg. of U specializing to $U(\mathfrak{g}^\theta)$

Q: Up to alg. iso., which subalgebras of U satisfy (i)-(iii)?

A: Standard analogues of $U(\mathfrak{g}^\theta)$.

(d) Standard Analogues

• Assume $\mathfrak{g}$ simple.

• $c \in A := \mathbb{C}[q, q^{-1}]_{(q-1)}$ (localized at); c specializes to 1 as $q \to 1$.

• $\tilde{\theta}_c : U \to U$ alg. iso: agrees with $\tilde{\theta}$ on all generators except on x_r, y_r.

$$\tilde{\theta}_c(x_r) = c\,\tilde{\theta}(x_r)$$
$$\tilde{\theta}_c(y_r) = c^{-1}\tilde{\theta}(y_r)$$

Note: $\tilde{\theta} \rightsquigarrow B$

$\tilde{\theta}_c \rightsquigarrow B_{\tilde{\theta}_c}$ ← standard analogue of $U(\mathfrak{g}^\theta)$.

Theorem A subalg. of $\mathcal{U}$ satisfies (i)–(iii) if and only if it is isomorphic as an algebra to $B_{\tilde{\theta}_c}$ for some c.

Recall: $R[G]$, $\underbrace{R[G/K]}_{\text{symmetric space}}$ regular functions on G, G/K resp.

Note: $R[G/K] = R[G]^K = R[G]^{\mathcal{U}(g^\theta)} \subseteq R[G]$ (*)

right invariants (dualizing to functions the) left action of K

right action

Goal: Use B to create a quantum counterpart of (*).

Consider

$$R_q[G/K] := R_q[G]^B_r \subseteq R_q[G] \quad (**)$$

$\leftarrow$ B-invariants on right

Q: In what sense is (**) a quantum counterpart of (*)?

Theorem: left action

(i) $\mathcal{U}(g) \circlearrowright R[G/K]$ semisimple

[$\lambda \in P^+ \rightsquigarrow m_\lambda$ = the multiplicity of the irrep. of highest weight λ]

(ii) Assume q is not a root of unity.

$\mathcal{U} \circlearrowright R_q[G/K]$ semisimple.

[$\lambda \in P^+$, M_λ = the multiplicity of the irrep. of highest weight λ]

(iii) $\forall\ \lambda \in P^+$, $m_\lambda = M_\lambda$ (the only nonzero value they can take is 1)