GRT Learning Seminar

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Quantum Kac-Moody symmetric pairs II

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Outline:

0) Recap
1) Affine Lie algebras
2) Quantum Loop algebras
3) Some philosophy: YBE and RE (Yang-Baxter equation) (reflection equation)
4) Case of $\mathcal{U}_q(\widehat{\mathfrak{gl}}_n)$

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0) Recap:

$$\boxed{\dim(\theta(\mathfrak{b}^+)\cap\mathfrak{b}^+)<\infty}$$

- $\mathfrak{g}=\mathfrak{g}(A)$ Kac-Moody algebra
- Admissable pair $(X,\tau) \rightsquigarrow$ involution $\theta=\theta(X,\tau)$ (of 2^{nd} kind)

  $X\subseteq I$ (finite type) $\quad \tau\in \mathrm{Aut}(A,X)$

- Lift to $\theta_q\in \mathrm{Aut}\,\mathcal{U}_q(\mathfrak{g}')$ $\rightsquigarrow$ Invariants $\mathcal{U}(\mathfrak{g}^\theta)$ $\rightsquigarrow$ right coideal subalg. $B_{c,s}$ specializing to $\mathcal{U}((\mathfrak{g}')^\theta)$ as $q\to 1$.

Some remarks:

1) Why involutions of 2^{nd} kind?

   Recall Letzter's assumptions (see Peter's talk)

   (a) $\theta(\mathfrak{h})=\mathfrak{h}$

   (b) $\theta|_{\mathfrak{g}_X}=\mathrm{Id}_{\mathfrak{g}_X}$

   (c) If $i\in I\setminus X$, then $\theta(e_i)\in\mathfrak{n}^-$, $\theta(f_i)\in\mathfrak{n}^+$

Lemma:

a) For general Kac-Moody, (a),(b),(c) $\Rightarrow$ 2^{nd} kind

b) For (X,τ) admissable, $\theta(X,\tau)$ satisfies (a),(b),(c).

2) θ_q is <u>not</u> generally an involution.
(unless restricted to $\langle e_i, f_i \mid i \in X \rangle$ subalg.)

3) $B_{c,s}$ isn't really $U_q(\mathfrak{g}')^{\theta_q}$ possibly also Cartan

1) Affine Lie algebras

- Assume that $\mathfrak{g}$ is f.d. simple $/\mathbb{C}$, $I = \{1, 2, \ldots, n\}$ Dynkin diagram and $\theta = \theta(X, \tau)$ an involution for $\mathfrak{g}$.
- Consider associated affine Lie algebra:

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$$
$$\hat{I} = \{0, 1, \ldots, n\} \supseteq I$$

- Consider $\hat{\theta} : \hat{\mathfrak{g}} \longrightarrow \hat{\mathfrak{g}}$ defined by:

$$\hat{\theta}(x \otimes t^k) = \theta(x) \otimes t^{-k}$$
$$\hat{\theta}(c) = -c \qquad \hat{\theta}(d) = -d$$

this is an involution of 2nd kind

<u>Lemma:</u>

a) $\exists\, \hat{\tau} \in \mathrm{Aut}(\hat{I}, X)$ by $\hat{\tau}(0) = 0$, $\hat{\tau}(i) = \tau(i) \ \forall i \neq 0$

b) $\hat{\theta} = \theta(X, \hat{\tau})$

- Have $\hat{\mathfrak{g}}' = [\hat{\mathfrak{g}}, \hat{\mathfrak{g}}] = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \circlearrowleft \hat{\theta}$

Denote $L(\mathfrak{g}) = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \circlearrowleft \hat{\theta}$

Then $\boxed{U(\hat{\mathfrak{g}}')/\langle c-1 \rangle \cong U(L\mathfrak{g})}$

<u>Lemma:</u> $L(\mathfrak{g})^{\hat{\theta}} = (\hat{\mathfrak{g}}')^{\hat{\theta}}$

2) Quantum loop algebras

- $\mathcal{U}_q(\hat{\mathfrak{g}}')$ contains a central element $\mathcal{C}$ which lifts c. (expressed in K's)

<u>Definition</u> The <u>quantum loop algebra</u> is

$$\mathcal{U}_q(L\mathfrak{g}) = \mathcal{U}_q(\hat{\mathfrak{g}}') \big/ \langle \mathcal{C} - 1 \rangle$$

2-sided ideal

Denote $\pi : \mathcal{U}_q(\hat{\mathfrak{g}}') \longrightarrow \mathcal{U}_q(L\mathfrak{g})$

- $\mathcal{U}_q(L\mathfrak{g})$ is a Hopf algebra, deforms $\mathcal{U}(L\mathfrak{g})$.
- $\mathcal{U}_q(L\mathfrak{g})$ has several presentations:
  - (i) Kac-Moody/Drinfeld-Jimbo type (in terms of Chevalley generators)] both for $\mathcal{U}_q(\hat{\mathfrak{g}}')$
  - (ii) Drinfeld's loop presentation ← most common
    $\mathcal{H}_{i,r}, \mathcal{E}_{i,r}, \mathcal{F}_{i,r} \rightsquigarrow h_i \otimes t^r$ etc.
  - (iii) RTT presentation (in classical types)
    depends on the type
- Related to the Yangian $Y(\mathfrak{g}) \rightsquigarrow$ q-Yangian

  not really a Yangian at all... ← $Y_q(\mathfrak{g}) = \langle$ modes with $r \geq 0$ $\rangle$
  i.e. generators

<u>Theorem</u> (Kolb)

Let (X, τ) be admissable for $\mathfrak{g}$, $\hat{\theta}$ as above, $\hat{\theta}_q$...
Choose c, s "specializable". Then:

a) $\pi(B_{c,s}) \cong B_{c,s}$

b) $\pi(B_{c,s})$ specializes to $\mathcal{U}(L\mathfrak{g})^{\hat{\theta}}$.

3) Philosophy

a) RTT formalism

(i) • Quantum group $\mathcal{U}$ ⇝ "should be" universal R-matrix $\mathcal{R} \in \widehat{\mathcal{U} \otimes \mathcal{U}}$ such that

$$\mathcal{R} \circ \Delta = \Delta^{op} \circ \mathcal{R}$$

• If $V_1, V_2 \in \mathrm{Rep}\,\mathcal{U}$, consider image $R_{V_1,V_2} \in \mathrm{End}(V_1 \otimes V_2)$

$$V_1 \otimes V_2 \xrightarrow{\text{twist} \circ R_{V_1,V_2}} V_2 \otimes V_1$$

• $R = R_{V,V}$ satisfies YBE $R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$ in $\mathrm{End}(V^{\otimes 3})$.

• Identify $V \cong \mathbb{C}^n$, can define an algebra $\mathcal{U}_V^{RTT}$

<u>generators:</u> t_{ij} $\quad 1 \leq i,j \leq n$

<u>relations:</u> $R \cdot T_1 \otimes T_2 = T_2 \otimes T_1 \cdot R \in \mathrm{End}(V^{\otimes 2}) \otimes \mathcal{U}_V^{RTT}$

where $T_1 = \sum e_{ij} \otimes I \otimes t_{ij}$ $\quad R = \sum r_{ijk\ell}\, e_{ij} \otimes e_{k\ell} \otimes 1$

$T_2 = \sum I \otimes e_{ij} \otimes t_{ij}$

<u>Expect:</u> If V is "nice enough", $\mathcal{U}$ is a quotient of $\mathcal{U}_V^{RTT}$
(faithful, etc.)

(ii) Reverse this process:

$R \in \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$ satisfying YBE ⇝ Hopf algebra $\mathcal{U}$

b) Reflection equation

<u>Q:</u> What is the RTT analogue of a coideal subalg. B?

(i)
- Coideal subalg. $B \subset \mathcal{U}$ $\rightsquigarrow$ universal K-matrix $\mathcal{K} \in \hat{\mathcal{U}}$ for B
- $V \in \mathrm{Rep}(\mathcal{U})$ $\rightsquigarrow$ image $K_V \in \mathrm{End}(V)$

satisfies: "reflection equation"

$$R_{V_2,V_1} \circ K_{V_2} \circ R_{V_1,V_2} \circ K_{V_1} = K_{V_1} \circ R_{V_2,V_1} \circ K_{V_2} \circ R_{V_1,V_2}$$

- K commutes with B. Denote $\rho: \mathcal{U} \longrightarrow \mathrm{End}(V)$

$$K_V \rho(b) = \rho^{\text{twisted}}(b) \cdot K_V \qquad V \xrightarrow{K} V^{\text{twisted}}$$

Remark: If V, V^{twisted} is irreducible then K_V is unique by Schur's lemma.

(ii) Reverse? Is B determined by $\mathcal{K}$?

"Physics": YBE $\longleftrightarrow$ scattering of 3 particles

RE $\longleftrightarrow$ interaction of 2 particles + a wall

Reference: Balagovic – Kolb "Universal K-matrix for quantum symmetric pairs"

$B_{c,s}$ $\rightsquigarrow$ $\mathcal{K}$

for certain c, s

4) Case of $U_q(\hat{gl}_n)$

- Extend θ from sl_n to gl_n by $\theta(I) = -I$
- R-matrix with "spectral parameters"

$$R_{u,v} = (u-v)\sum_{i\neq j} e_{ii}\otimes e_{jj} + (q^{-1}u - qv)\sum_i e_{ii}\otimes e_{ii}$$
$$+ (q^{-1}-q)u\sum_{i>j} e_{ij}\otimes e_{ji} + (q^{-1}-q)v\sum_{i<j} e_{ij}\otimes e_{ji}$$

satisfies YBE with spectral parameters.

- Associated RTT algebra is $\mathcal{U}_q(\widehat{\mathfrak{gl}}_n)$ $\rightsquigarrow$ Kolb
  $\rightsquigarrow$ Molev-Ragoucy-Sorba

<u>Proposition</u> (Frenkel-Mukhin)

$\exists$ an embedding of Hopf algebras:

$$\mathcal{U}_q(L\mathfrak{sl}_n) \hookrightarrow \mathcal{U}_q(\widehat{\mathfrak{gl}}_n)$$

$\boxed{\mathcal{U}_q(\widehat{\mathfrak{gl}}_n)}$

<u>generators</u>: $t_{ij}^{(r)}, \bar{t}_{ij}^{(r)}$ $\quad r \geq 0$ $\rightsquigarrow$ formal series

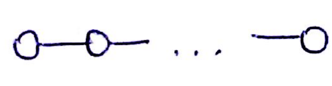
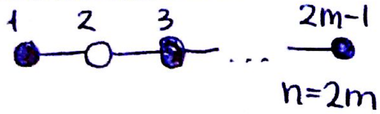
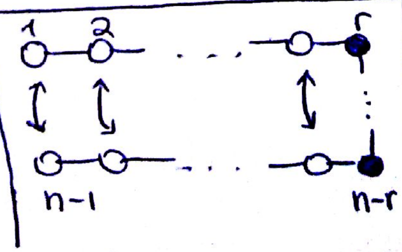
$(e_{ij}\otimes t^r \quad e_{ij}\otimes t^{-r})$ $\qquad t_{ij}(u) = \sum_{r\geq 0} t_{ij}^{(r)} u^{-r}$

<u>relations</u>:

$R(u,v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u,v)$ $\rightsquigarrow$ $T(u) = \sum_{i,j} e_{ij}\otimes t_{ij}(u)$

in $\mathrm{End}(\mathbb{C}^n\otimes\mathbb{C}^n)\otimes\mathcal{U}_q(\widehat{\mathfrak{gl}}_n)((u^{-1},v^{-1}))$ $\qquad \in \mathrm{End}(\mathbb{C}^n)\otimes\mathcal{U}_q(\widehat{\mathfrak{gl}}_n)[[u^{-1}]]$

- For $\mathfrak{sl}_n$ the admissable pairs / Satake diagrams are:

type	Satake diag.	(X,τ)	$\mathfrak{g}^\theta$
$A_n 1$		$(\emptyset, \mathrm{Id}_I)$	$\mathfrak{so}_n$
$A_n 2$		$(\{1,3,\dots,2m-1\}, \mathrm{Id}_I)$	$\mathfrak{sp}_{2m}$
$A_n 3/4$		$(\{r,r+1,\dots,n-r\}, \tau_I)$	$\mathfrak{sl}_n \cap (\mathfrak{gl}_r \oplus \mathfrak{gl}_{n-r})$

- For A_n1, A_n2 coideal subalg. described in RTT by MRS:

Let $K = \begin{cases} I_{n\times n} & \text{Case A 1} \\ q\sum_{k=1}^{m} e_{2k-1,2k} - \sum_{k=1}^{m} e_{2k,2k-1} & \text{case A 2} \end{cases}$

<u>Definition</u> The subalgebra generated by all coeff's of $T(u)K\overline{T}(u^{-1})^t \subset U_q(\widehat{gl_n})$ is called $Y_q^{tw}(O_n)$ (resp. $Y_q^{tw}(Sp_{2m})$). "twisted q-Yangians".

<u>Theorem</u> (MRS)
In each case, as $q \to 1$, $Y_q^{tw} \longrightarrow \mathcal{U}(L(gl_n)^{\hat{\theta}})$.

<u>Proposition</u> (MRS)
In each case, K solves RE for Y_q^{tw}.

<u>Theorem</u> (MRS)
Y_q^{tw} is a coideal subalgebra. motivated by integrable systems (?)

<u>Thm</u> (Kolb)
By choosing $s = 0$, and choosing c specially
$$B_{c,s} \cong Y_q^{tw} \cap U_q(Lsl_n) \subseteq U_q(\widehat{gl}_n)$$

<u>Remarks</u>

1) Does $T(u)K(u)\overline{T}(u^{-1})^t$ give B?
2) Regelskis, Vlaar (in progress) write down explicit K-matrices in many classical types.