Quantum Symmetric Kac-Moody Pairs I

(Alex Weekes)

Ref: Kolb

Plan: 1) Kac-Moody symmetric pairs and automorphisms
2) Lifting to the quantum group
3) Quantum symmetric Kac-Moody pairs

Next time: loop version ⇝ reflection equation, q-Yangians

Part I

Notation: generalized Cartan matrix

- k alg. closed field, char 0
- Let $A = (a_{ij})$ be a GCM, assume symmetrizable & indecomposable

$a_{ij} \in \mathbb{Z}$ $\qquad a_{ij} \leq 0$ for $i \neq j$

$a_{ii} = 2$ $\qquad a_{ij} = 0 \Rightarrow a_{ji} = 0$

- $\mathfrak{g} = \mathfrak{g}(A)$ assoc. K-M. Lie alg. / k.

$\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}] = \langle e_i, f_i, h_i \rangle$ Chevalley generators

Let $\omega : \mathfrak{g} \to \mathfrak{g}$ be the Chevalley involution $e_i \mapsto -f_i$, $f_i \mapsto -e_i$, $h \mapsto -h$

- $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, $\mathfrak{b}^+, \mathfrak{b}^-$
- Roots $\Delta \supset \Pi = \{\alpha_i\}_{i \in I}$

$Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ root lattice

involution $\theta : \mathfrak{g} \to \mathfrak{g}$ ⇝ $\mathfrak{g}^\theta$

- $W = \langle s_i \rangle$ Weyl group

1) Automorphisms of $\mathfrak{g}$

Def. An element $\sigma \in \mathrm{Aut}(\mathfrak{g})$ is called

(i) <u>1st type</u> if $\dim(\sigma(\mathfrak{b}^+) \cap \mathfrak{b}^-) < \infty$ $\rightsquigarrow$ deg 0

(ii) <u>2nd type</u> if $\dim(\sigma(\mathfrak{b}^+) \cap \mathfrak{b}^+) < \infty$ $\rightsquigarrow$ deg 1

* This gives a "$\mathbb{Z}_2$-grading" of $\mathrm{Aut}(\mathfrak{g})$, 1st type is even, 2nd type is odd

not f.d. $\boxed{\mathrm{Aut}(\mathfrak{g}) = \mathrm{Aut}(\mathfrak{g})_{\text{even}} \cup \mathrm{Aut}(\mathfrak{g})_{\text{odd}}}$

<u>Note</u>: If $\mathfrak{g}$ is f.d., everything is both 1st and 2nd.

<u>Example</u> ω is 2nd type $\omega(\mathfrak{n}^+) = \mathfrak{n}^-$

$\omega(\mathfrak{n}^-) = \mathfrak{n}^+$

<u>Some subsets of $\mathrm{Aut}(\mathfrak{g})$</u>

a) Let G be the (adjoint) group for $\mathfrak{g}$, and

$$\mathrm{Ad}(G) = \{\mathrm{Ad}_g \mid g \in G\} \subseteq \mathrm{Aut}(\mathfrak{g})$$

$\boxed{\text{all of } 1^{st} \text{ kind}}$

"inner aut's" $= \{\exp(\mathrm{ad}_x) \mid x \in \mathfrak{g}_\alpha,\ \alpha \in \Delta^{re}\}$

real roots

G contains

$$\tilde{W} = \langle \tilde{s}_i \rangle, \quad \tilde{s}_i = \exp(e_i)\exp(-f_i)\exp(e_i)$$

($\mathbb{Z}_2$ cover of the Weyl group)

these satisfy the braid relations (but don't square to 0)

$\rightsquigarrow$ can define $\tilde{w} \in \tilde{W}$ for $w \in W$

b) Let $\tilde{H} = \mathrm{Hom}_{\text{groups}}(Q, k^\times)$

For $x \in \tilde{H}$, define $\mathrm{Ad}(x) \in \mathrm{Aut}(\mathfrak{g})$ by $\boxed{1^{st} \text{ kind}}$

$$\mathrm{Ad}|_{\mathfrak{h}} = \mathrm{Id}_{\mathfrak{h}}, \quad \mathrm{Ad}(x)(e_\alpha) = x(\alpha) \cdot e_\alpha \quad \alpha \in \Delta$$

(preserve $\mathfrak{b}^+$ & $\mathfrak{b}^-$) "rescaling"

c) Consider $\mathrm{Aut}(A)$ = (graph) automorphisms of the Dynkin diagram

$$= \left\{ \begin{array}{c} \text{bijections} \\ \tau : I \to I \end{array} \,\middle|\, a_{ij} = a_{\tau(i)\tau(j)} \right\}$$

Acts naturally on $\mathfrak{g}'$, $\tau(X_i) = X_{\tau(i)}$ $\forall\ X = e, h, f$

$\boxed{\text{Extends to } \mathfrak{g}}$

Let $\mathrm{Out}(A) = \underbrace{\mathrm{Aut}(A)}_{\boxed{1^{st}\text{ kind}}} \cup \underbrace{\omega \cdot \mathrm{Aut}(A)}_{\boxed{2^{nd}\text{ kind}}}$

"outer aut's"

preserve triangular decomp.

d) Let $\mathrm{Aut}(\mathfrak{g}, \mathfrak{g}') = \{\sigma \in \mathrm{Aut}(\mathfrak{g}) \mid \sigma|_{\mathfrak{g}'} = \mathrm{Id}\}$ $\boxed{1^{st}\text{ kind}}$

<u>Theorem</u> (Kac-Wang '92)

$$\mathrm{Aut}(\mathfrak{g}) = \mathrm{Out}(A) \ltimes \left(\mathrm{Aut}(\mathfrak{g}, \mathfrak{g}') \times \left(\mathrm{Ad}(\tilde{H}) \ltimes \mathrm{Ad}(G)\right)\right)$$

<u>Corollary</u> Any $\sigma \in \mathrm{Aut}(\mathfrak{g})$ of 2^{nd} type can be written as:

$$\sigma = \underset{\mathrm{Ad}(\tilde{H})}{\underset{\cap}{\mathrm{Ad}(x)}} \circ \underset{\mathrm{Aut}(\mathfrak{g},\mathfrak{g}')}{\underset{\cap}{f}} \circ \tau \circ \omega \circ \mathrm{Ad}(g), \quad \tau \in \mathrm{Out}(A), \quad \mathrm{Ad}(g) \in \mathrm{Ad}(G).$$

2) <u>Admissible pairs of $\mathfrak{g}$</u>

• Let $X \subseteq I$ be of <u>finite type</u> $\rightsquigarrow W_X \subseteq W$, $W_X = \langle s_i \mid i \in X \rangle$

$\rightsquigarrow \mathfrak{g}_X \subseteq \mathfrak{g}$, $\mathfrak{g}_X = \langle e_i, f_i \mid i \in X \rangle$ — subroot system corr. to X

• Denote $\mathrm{Aut}(A, X) = \{\tau \in \mathrm{Aut}(A) \mid \tau(X) = X\}$

[<u>Example</u> There is one τ_X corresponding to w_X = longest elt. of W_X
Really, for $i \in X$, $\alpha_i \mapsto -w_X(\alpha_i) = \alpha_{\tau_X(i)}$]

<u>Def.</u> A pair (X, τ) where $X \subseteq I$, $\tau \in \mathrm{Aut}(A, X)$ is called <u>admissable</u> if

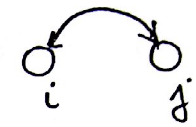
(i) τ is an involution

(ii) $\tau|_X = \tau_X$

(iii) For $j \in I \setminus X$ with $\tau(j) = j$, then $\langle \alpha_j, \rho^\vee_X \rangle \in \mathbb{Z}$

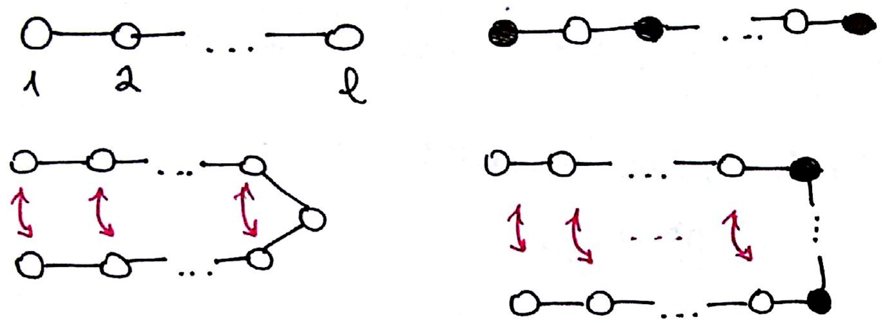
• For $\mathfrak{g}$ of <u>finite type</u> (I, τ_I) is always admissable.

All other admissable pairs correspond to Satake diagrams.

- 2-colouring / marking of Dynkin diagram
- nodes of X are coloured solid $\bullet$
- If $i, j \in I \setminus X$ and $\tau(i) = j$, then 

Ref: Araki $\rightsquigarrow$ correspond to real forms for $\mathfrak{g}$.

Example In type A,



Goal: Associate an involution to every admissable pair.

Theorem (Kolb) Let (X, τ) be admissable. Then

$$\theta(X,\tau) = \mathrm{Ad}(\underset{\substack{\cap \\ \tilde{H}}}{s(X,\tau)}) \circ \tau \circ \omega \circ \mathrm{Ad}(\tilde{w}_X) \in \mathrm{Aut}(\mathfrak{g})$$

$\hookrightarrow$ depends on a choice of ordering

is an involution of 2nd kind. of the Dynkin diagram

Main Theorem (Kolb)

There is a bijection $(X,\tau) \mapsto \theta(X,\tau)$

$$\left\{ \begin{matrix} \text{Admissable} \\ \text{pairs} \end{matrix} \right\} \Big/ \mathrm{Aut}(A) \longleftrightarrow \left\{ \begin{matrix} \text{Involutions} \\ \theta \in \mathrm{Aut}(\mathfrak{g}) \\ \text{of } 2^{\text{nd}} \text{ kind} \end{matrix} \right\} \Big/ \mathrm{Aut}(\mathfrak{g})$$

Lemma For $\theta = \theta(X,\tau)$, $\mathfrak{g}^\theta$ is generated by

a) e_i, f_i, h_i for $i \in X$

b) $[h$ for $h \in \mathfrak{h}^\theta]$ $\Longleftrightarrow$ h_α for $\alpha \in Q$, $\tau_X(\alpha) = \alpha$ — true that $\theta(\mathfrak{h}) = \mathfrak{h}$

c) $f_i + \theta(f_i)$ for $i \in I \setminus X$

$\left(\begin{matrix} \tau_X(\alpha_i) = \alpha_{\tau_X(i)} \\ \text{+ extend linearly} \end{matrix} \right)$

Remark

1) For $U(\mathfrak{g}^\theta)$, can choose $s = (s_i) \in k^{I \setminus X}$. Then $U(\mathfrak{g}^\theta)$ is generated by a), b) and c') $f_i + \theta(f_i) + s_i$, $i \in I \setminus X$

2) Similar holds for $(\mathfrak{g}')^\theta$.

Part II: Quantum case

1) Quantum groups and their automorphisms

- Consider $\mathcal{U} = U_q(\mathfrak{g}') = \langle E_i, F_i, K_i^{\pm 1} \rangle$ over $k(q)$. "$K_i = q^{h_i}$"
  ↳ Hopf subalg. of $U_q(\mathfrak{g})$ $\left(\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i \right)$
- Chevalley involution extends
  ↳ not HA autom.

$$\omega: E_i \longleftrightarrow -F_i$$
$$K_i \longleftrightarrow K_i^{-1}$$

one of the generators

- $\mathrm{Aut}(A) \ni \tau \rightsquigarrow \tau(X_i) = X_{\tau(i)}$ $\qquad X = E, F, K^{\pm 1}$
  ↳ is a HA autom.

- Set $\tilde{H}_q = \mathrm{Hom}_{gps}(Q, k(q)^\times)$

$x \in \tilde{H}_q$ $\quad \mathrm{Ad}(x)$:
$$E_i \mapsto x(\alpha_i) E_i$$
$$F_i \mapsto x(\alpha_i)^{-1} F_i$$
$$K_i \mapsto K_i$$

Theorem (Twietmeyer)

$$\mathrm{Aut}_{Hopf}(\mathcal{U}) \cong \tilde{H}_q \rtimes \mathrm{Aut}(A)$$

Remark This is partial evidence for why lifts of $U(\mathfrak{g}^\theta)$ should <u>not</u> be Hopf subalg's of $\mathcal{U}$.

- Have Lusztig's automorphisms T_i, $i \in I$, of $\mathcal{U}$. These satisfy braid relations $\rightsquigarrow$ can define T_w $\forall w \in W$

• Define also $\Psi : \mathcal{U} \to \mathcal{U}$ $\quad \Psi(E_i) = E_i K_i$

$\Psi(F_i) = K_i^{-1} F_i$

$\Psi(K_i) = K_i$

Def: Let (X, τ) be admissable.
Set $\Theta_q(X,\tau) = \mathrm{Ad}(s(X,\tau)) \circ T_{w_X} \circ \Psi \circ \tau \circ \omega$

2) Quantum Kac-Moody symmetric pairs

• Choose $s = (s_i) \in k(q)^{I \setminus X}$ $\qquad$ standard: all $s_i = 0$

$c = (c_i) \in (k(q)^\times)^{I \setminus X}$

Def: Let $B_{c,s} \subseteq \mathcal{U}$ be generated by

a) $E_i, F_i, K_i^{\pm 1}$ $(K_i = K_{\alpha_i})$ for $i \in X$

b) K_α $\qquad$ for $t_X(\alpha) = \alpha$, $\alpha \in Q$

c) $B_i = F_i + c_i \Theta_q(F_i K_i) \cdot K_i^{-1} + s_i$

$\boxed{\begin{array}{l} Q \to T \\ \alpha_i \mapsto K_i \end{array}}$

Remark $\underset{\{ q \to 1}{} f_i + \Theta(f_i) + s_i$ if $c_i \xrightarrow[q \to 1]{} 1$

Proposition $B_{c,s}$ is a right coideal subalgebra.
i.e. $\Delta(B_{c,s}) \subseteq \mathcal{U} \otimes B_{c,s}$

Def Adding some extra conditions on c, s, (related to how c_i and $c_{\tau(i)}$ are related)
$B_{c,s}$ is called quantum symmetric pair coideal subalgebra.

3) Specialization

• Let $A = k[q]_{(q-1)}$, assume $c_i \in A$ and $c_i \xrightarrow{q \to 1} 1$

Theorem

1) $\mathcal{U} \otimes_A k \xrightarrow{\sim} U(\mathfrak{g}')$

2) $\Theta_q(X,\tau)$ specializes to $\Theta(X,\tau)$

3) $B_{c,s}$ specializes to $U((\mathfrak{g}')^\Theta)$ (Kolb)

i.e.

$$\begin{tikzcd} \mathcal{U} \arrow[r, "\theta_q"] \arrow[d, "q\to 1"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{U} \arrow[d, "q\to 1"] \\ U(\mathfrak{g}') \arrow[r, "\theta"] & U(\mathfrak{g}') \end{tikzcd}$$

<u>Theorem</u> If $W \subseteq \mathcal{U}$ is a subspace containing $B_{c,s}$ and $W \xrightarrow[q\to 1]{} \mathcal{U}((\mathfrak{g}')^{\theta})$, then $W = B_{c,s}$.

<u>Remark</u>: In Kolb's story, can say <u>explicitly</u> what relations in $B_{c,s}$ are.