Theorem (i) $\mathcal{U}(\mathfrak{g}) \overset{\text{left}}{\circlearrowright} R[G/K]$. semisimple

$[\lambda \in P^+ \rightsquigarrow m_\lambda$ multiplicity of irrep $V(\lambda)]$

(ii) Assume q is not a root of unity.

$\mathcal{U} \overset{\text{left}}{\circlearrowright} R_q[G/K]$ semisimple

$[\lambda \in P^+ \rightsquigarrow M_\lambda$ mult. of irrep $V(\lambda)$ of $\mathcal{U}]$

(iii) For all $\lambda \in P^+$, $m_\lambda = M_\lambda$.

Lie superalgebras : An Intro — Khoa

Ref Kac's paper

Musson's book.

Carmeli – Caston – Fioresi : Math. Foundation of supersymmetry (Appendix)

I Definition / Examples.

- Work over $\mathbb{C}$
- A superalgebra is a $\mathbb{Z}/2\mathbb{Z}$-graded vect space $A = A_0 \oplus A_1$ s.t. $A_\alpha \cdot A_\beta \subset A_{\alpha+\beta}$ $\alpha, \beta \in \mathbb{Z}/2\mathbb{Z}$ and the product is bilinear.
- A Lie superalgebra is a super algebra with product $[\cdot, \cdot]$ s.t.

(1). $[a,b] = -(-1)^{\bar{a}\bar{b}}[b,a]$ a,b homogeneous, $\bar{x} = \deg(x)$

(2). $(-1)^{\bar{a}\bar{c}}[a,[b,c]] + (-1)^{\bar{b}\bar{a}}[b,[c,a]] + (-1)^{\bar{c}\bar{b}}[c,[a,b]] = 0$.

<u>Note</u> 1). $\mathfrak{g}_0$ is a Lie algebra.

$\mathfrak{g}_1$ is a $\mathfrak{g}_0$-module

2). One can define subalg, ideals in the usual way, (but assume $\mathbb{Z}_2$-graded).

3). $\varphi : \mathfrak{g} \to \mathfrak{g}'$ is a Lie superalg hom. if

$\varphi([a,b]) = [\varphi(a), \varphi(b)]$, $\varphi(\mathfrak{g}_i) \subseteq \mathfrak{g}'_i$ "even map"

<u>Examples</u> 0). Any Lie alg $\mathfrak{g}_0 = \mathfrak{g}_0 \oplus 0$.

1). A superalgebra (associative)

$[a,b] := ab - (-1)^{\bar{a}\bar{b}} ba$ a, b homogeneous.

with same grading.

2). $V = V_0 \oplus V_1$ $\mathbb{Z}_2$-graded vect space.

$\mathrm{End}(V) = \mathrm{End}(V)_0 \oplus \mathrm{End}(V)_1$

$\mathrm{End}(V)_k = \{X \in \mathrm{End}(V) \mid X(V_i) \subseteq V_{i+k}\}$ $\rightsquigarrow$ assoc. superalg.

$\overset{(1)}{\rightsquigarrow}$ Lie superalg.

$$\overset{m}{\underset{|}{}} \quad \overset{n}{\underset{|}{}}$$

Pick a basis for V_0, V_1

$$\mathfrak{g} = \mathfrak{gl}(m,n) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \;\middle|\; A \ m\times m,\ D \ n\times n \right\}.$$

$$\mathfrak{g}_0 = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \quad \mathfrak{g}_1 = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}.$$

(B). $A(m,n)$: For $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{gl}(m,n)$

$$\mathrm{Str}(X) = \mathrm{tr}(A) - \mathrm{tr}(D)$$

<u>Properties</u> $\mathrm{Str}(XY) = 0$ if $\bar{X} = 0$, $\bar{Y} = 1$

$\mathrm{Str}(XY) = (-1)^{\bar{X}\bar{Y}}\ \mathrm{Str}(YX)$.

$\mathrm{Str}([X,Y]Z) = \mathrm{Str}(X[Y,Z])$.

$\mathrm{Str}([X,Y]) = 0$.

$$\mathfrak{g} = \mathfrak{sl}(m,n) = \{ X \in \mathfrak{gl}(m,n) \mid \mathrm{Str}(X) = 0 \}$$

$$\Rightarrow \mathfrak{g}_0 = \left(\begin{array}{c|c} \mathfrak{sl}_m & 0 \\ \hline 0 & 0 \end{array}\right) \oplus \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \mathfrak{sl}_n \end{array}\right) \oplus \mathbb{C}\underbrace{\left(\begin{array}{c|c} nI_m & 0 \\ \hline 0 & mI_n \end{array}\right)}_{\substack{\shortparallel \\ I_{m,n}}}$$

$$\mathfrak{g}_1 = \left(\begin{array}{c|c} 0 & B \\ \hline 0 & 0 \end{array}\right) \oplus \left(\begin{array}{c|c} 0 & 0 \\ \hline C & 0 \end{array}\right)$$

$$A(m,n) = \begin{cases} \mathfrak{sl}(m+1, n+1) & \text{if } m \neq n. \\ \dfrac{\mathfrak{sl}(n+1, n+1)}{\mathbb{C}\, I_{n+1,n+1}} & \text{if } m = n. \end{cases}$$

(4) Orthosymplectic

$V = V_0 \oplus V_1$ $\qquad$ b: bilinear form on V

$b|_{V_0}$ is symm. , $b|_{V_1}$ is skew-symm. $\quad (\dim V_1 = 2r)$

$V_0 \perp_b V_1$.

$osp(m,n)_i = \{ X \in gl(m,n)_i \mid b(X(v), w) + (-1)^{\bar{j}\bar{i}} b(v, Xw) = 0 \}$

subalg of $gl(m,n)$

Pick convenient basis for V

$m = 2l+1$; $\quad g_0 \cong B_l \oplus C_r \quad g_1 \cong so_m \otimes sp_n$. (?)

$m = 2l \ (l \geq 2) \quad g_0 \cong D_l \oplus C_r$, $g_1 \cong so_m \otimes sp_n$

$m = 2$ (can also be done).

$B(m,n) = osp(2m+1, 2n) \qquad m \geq 0, n > 0$

$D(m,n) = osp(2m, 2n) \qquad m \geq 2, n > 0$

$C(n) = osp(2, 2n-2) \qquad (n \geq 2)$.

(5) Others. $\quad$ Strange $p(n)$, $q(n)$

$D(2,1,\alpha)$ $(\alpha \in \mathbb{C} \setminus \{0,-1\})$ $\quad$ 17-dim

$F(4)$ $\quad$ 40-dim.

$G(3)$ $\quad$ 31-dim

The above Lie superalgebras are classical.

(6). A : assoc. superalgebra

$$\mathrm{Der}(A)_k = \{ D \in \mathrm{End}(A)_k \mid D(ab) = D(a)b + (-1)^{k\bar{a}} a D(b) \}$$

$$\mathrm{Der}(A) = \mathrm{Der}(A)_0 \oplus \mathrm{Der}(A)_1$$

$\Lambda(n) = \langle \xi_1, \cdots, \xi_n \rangle$ Exterior alg gen by $\xi_1, \cdots, \xi_n$.

$$W(n) = \mathrm{Der}(\Lambda(n)) = \bigoplus_{k \geq -1} W(n)_k.$$

where $W(n)_k = \{ D \in W(n) \mid D(W(n)_i) \subseteq W(n)_{i+k} \}$

$$= \{ \sum P_i \frac{\partial}{\partial \xi_i} \mid P_i \in \Lambda(n),\ \deg P_i = k+1 \}$$

"Cartan type"

$$\left(\frac{\partial}{\partial \xi_i} (\xi_j) = \delta_{ij} \right)$$

II. General result / concept :

1/ Universal enveloping alg. of $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$

By def, it is a pair $(\mathcal{U}, i)$ ($\mathcal{U}$ assoc. alg. w/ 1 not necessarily $\mathbb{Z}_2$-graded)

$i : \mathfrak{g} \to \mathcal{U}$

$$i([x,y]) = i(x)\,i(y) - (-1)^{\bar{x}\bar{y}}\, i(y)\,i(x).$$

with universal property.

More concretely,

$$\mathcal{U}\mathfrak{g} = T(\mathfrak{g}) \Big/ \langle x\otimes y - (-1)^{\bar{x}\bar{y}} y\otimes x - [x,y] \rangle$$

PBW thm for Lie superalg.

$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, $\beta_0 = \{a_1, \cdots, a_m\}$ basis of $\mathfrak{g}_0$

$\beta_1 = \{b_1, \cdots, b_n\}$ - - - - $\mathfrak{g}_1$

Then a basis for $\mathcal{U}\mathfrak{g}$ is

$a_1^{k_1} \cdots a_m^{k_m} b_{i_1} \cdots b_{i_s}$ $\quad k_i \geq 0$, $\quad 1 \leq i_1 < i_2 < \cdots < i_s \leq n$.

2/ Representations

A rep. of $\mathfrak{g}$ is a $\mathbb{Z}_2$-graded space $V = V_0 \oplus V_1$

w/ a hom : $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$.

more explicitly, $\mathfrak{g}_i V_j \subseteq V_{i+j}$

$$[x_1, x_2]v = x_1(x_2 \cdot v) - (-1)^{\bar{x}_1\bar{x}_2} x_2(x_1 v)$$

(can define submodules)

Homomorphisms of modules do not require preservation of grading

Schur's lemma $V = V_0 \oplus V_1$

$M \subseteq \mathfrak{gl}(V)$ acting irreducibly on V. Let

$C(M) = \{X \in \mathfrak{gl}(V) \mid [X, m] = 0 \ \forall m \in M\}$

Then either $C(M) = \text{span}\{Id\}$

or $C(M) = \text{Span}\{Id, A\}$.

where A nonsingular permuting V_0 and V_1 $(A^2 = Id)$

Lie's thm does NOT hold

E.g. $N = N_0 \oplus N_1 = \text{span}\{e\} \oplus \text{span}\{a_1, \cdots, a_n, b_1, \cdots, b_n\}$.

$[a_i, b_i] = e$

$\alpha \in \mathbb{C} \setminus \{0\}$. $\Lambda(n)$ becomes an N-module

$$\rho_\alpha(a_i)u = \frac{\partial u}{\partial \xi_i}, \quad \rho_\alpha(b_i)u = \alpha \xi_i u$$

$$\rho_\alpha(e)u = \alpha u.$$

Check ρ_α is an irred. N-module of dim 2^n

but N nilpotent.

Engel's thm still holds.

$\mathfrak{g}$ subalg $\subset \mathfrak{gl}(V)$, all elements of $\mathfrak{g}$

are nilpotent, then $\exists v \neq 0 \in V$ s.t. $xv = 0$ $\forall x \in \mathfrak{g}$.

Ado's thm. $\mathfrak{g} \hookrightarrow \mathfrak{gl}_n$ ✓

Semisimple $\neq$ direct sum of simples

↑

no nonzero abelian ideal

(§5 in Kac: construct semisimple from simple ones)

III Classification of simple Lie algebra

- simple Lie algebra : no nontrivial ideal.

- Two step : classical, nonclassical (Cartan type ?)

- A Lie superalg. $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is classical iff $\mathfrak{g}$ is simple and the $\mathfrak{g}_0$-mod $\mathfrak{g}_1$ is completely reducible.

- $\mathfrak{g}$ simple. Then any invariant bilinear form on $\mathfrak{g}$ is either non-degenerate or 0.

- $\mathfrak{g}$ classical $\Leftrightarrow$ $\mathfrak{g}_0$ is reductive.
  (and simple?)

Outline of classification of classical simple.

- nondeg. Killing form
  - $\mathfrak{g}_1$ irred ①
  - reducible ②
- 0
  - $\mathfrak{g}_1$ irred ③
  - reducible. ④

① : $\mathfrak{g} \cong B(m,n)$, $D(m,n)$ $(m-n \neq 1)$, $F(4)$, $G(3)$.

② $A(m,n)$ $(m \neq n)$, $C(n)$

③ $\mathfrak{q}(n)$, $D(n+1,n)$, $D(2,1;\alpha)$

④ $A(n,n)$, $\mathfrak{p}(n)$.

IV Root system : $\mathfrak{h}_0 \oplus \mathfrak{h}_1$

A Cartan subalg $\mathfrak{h} \subset \mathfrak{g}$ is nilpotent, self-normalizing subalgebra

If $\alpha \in \mathfrak{h}_0^*$, define $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h,x] = \alpha(h)x \ \forall h \in \mathfrak{h}_0\}$

α root if $\mathfrak{g}_\alpha \neq 0$. $\rightsquigarrow \Delta$

α even if $\mathfrak{g}_0 \cap \mathfrak{g}_\alpha \neq 0$ $\rightsquigarrow \Delta_0$

odd $\mathfrak{g}_1 \cap \mathfrak{g}_\alpha \neq 0$ $\rightsquigarrow \Delta_1$

$\Delta = \Delta_0 \cup \Delta_1$

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

$\mathfrak{g}$ is called "basic classical" if $\mathfrak{g}$ is simple, $\mathfrak{g}_0$ reductive, $\mathfrak{g}$ admits a nondeg consistent invariant bilinear form.

Can study root system of basic classical $\mathfrak{g}$

Π simple roots, not linear independent.

Example $A(m,n)$ $m \neq n$.

$\mathfrak{h} = \mathfrak{h}_0$ = subalg of diagonal matrices

$\varepsilon_i, \delta_i \in \mathfrak{h}^*$ $\begin{cases} \varepsilon_i(\mathrm{diag}(a_1, \cdots, a_{m+n+2})) = a_i \\ \delta_j(\mathrm{diag}(a_1, \cdots, a_{m+n+2})) = a_{m+1+j} \end{cases}$

Root system:

$\Delta_0 = \{\varepsilon_i - \varepsilon_j, \delta_k - \delta_l\}$

$\Delta_1 = \{\pm(\varepsilon_i - \delta_k)\}$ $\quad 1 \leq i \neq j \leq m+1$

$1 \leq k \neq l \leq n+1$

∇ A bit about representation theory

$l \subseteq \mathfrak{h}^*$ 1-dim line s.t. $l \cap \Delta \neq \phi$

$\mathfrak{g}^l$ = subalg gen. by $\mathfrak{g}_\alpha$ $\alpha \in l$

[Penkov - Serganova]

Then $\mathfrak{g}^l$ can be $\mathfrak{sl}_2$, $\mathfrak{osp}(1,2)$, $\mathfrak{sl}(1,1)$, $\mathfrak{q}(2)$

Rep theory of $\mathfrak{osp}(1,2)$.

$$\mathfrak{osp}(1,2) = \left\{ \begin{pmatrix} 0 & \alpha & \beta \\ -\beta & & \\ \alpha & & B \end{pmatrix} \;\middle|\; B \in \mathfrak{sl}_2 \right\} \subseteq \mathfrak{gl}(1,2)$$

$$e = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$X = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$\mathfrak{g}_0 = \text{span}\{e, f, h\} \cong sl_2$

$\mathfrak{g}_1 \cong \text{span}\{x, y$

<u>Thm</u> V fin. irrep of $osp(1,2)$. Then V has dim $2n+1$ and $\exists$ bases

$$\underbrace{v_0, \cdots, v_n}_{V_0}, \underbrace{w_0, \cdots, w_{n-1}}_{V_1}$$

$h v_i = (n-2i) v_i \qquad e v_i = (n-i+1) v_{i-1}, \qquad f v_i = (i+1) v_{i+1}$

$x v_i = w_{i-1} \qquad y v_i = w_i$

Similarly for w_i.. This classifies irrep of $osp(1,2)$.

$$\begin{tikzcd}
V_0 & \cdots & v_n \arrow[rr, shift left, "e"] \arrow[dr, shift right, "x"'] & & v_{n-1} \arrow[ll, shift left, "f"] \arrow[rr, shift left, "e"] \arrow[dl, shift right, "y"'] & & v_{n-2} \arrow[ll, shift left, "f"] & \cdots \\
 & \cdots & & & & \cdots & & \\
V_1 & \cdots & & w_{n-1} \arrow[uul, shift right, "y"'] \arrow[uur, shift right, "x"'] \arrow[rr, shift left, "e"] & & w_{n-2} \arrow[ll, shift left, "f"] & \cdots &
\end{tikzcd}$$